THE JOURNAL
OF THE
INDIAN MATHEMATICAL SOCIETY

Edited by

K. CHANDRASEKHARAN

with the co-operation of

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PAUL ERDÖS
V. GANAPATHY IYER
SAUNDERS MACLANE
S. MINAKSHISUNDARAM
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NEW SERIES

Vol. XVII

1953

PUBLISHED BY THE SOCIETY

Annual Subscription: Rs. 12

PRINTED AT THE
COMMERCIAL PRINTING PRESS
BOMBAY

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ON EIGENFUNCTIONS OF THE MEMBRANE PROBLEM

By P. SAMBASIVA RAO

[Received March 1, 1952 and, in revised form, September 5, 1952]

1. Introduction.

The main problem of the vibrating membrane may be stated thus. Let D be a bounded domain in the (x_1, x_2) plane bounded by a regular curve B. It is known that the boundary value problem

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \lambda u = 0, \tag{1.1}$$

$$u(x_1, x_2) = 0 \text{ on } B,$$

has solutions for a discrete set of positive values of λ, which are called the eigenvalues and the corresponding solutions the eigenfunctions. The study of the distribution of eigenvalues and the behaviour of eigenfunctions is the scope and purpose of the membrane problem. H. Weyl [10] and R. Courant [4] considered the problem of distribution of eigenvalues, and T. Carleman [3], R. Courant, A. Pleijel [7] and S. Minakshisundaram [6] studied the behaviour of the eigenfunctions.

Let D denote a bounded domain in the k-dimensional Euclidean space with a regular boundary B. Then the boundary value problems

$$\sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i^2} + \lambda u = 0,$$

$$\text{I.} \qquad u(x_1, x_2, \ldots x_k) = 0 \text{ on } B, \tag{1.2}$$

and

$$\text{II.} \qquad \frac{\partial u}{\partial n} = 0 \text{ on } B,$$

where $\frac{\partial}{\partial n}$ denotes the derivative along the normal to the boundary,

The substance of this paper formed part of the thesis approved for the D.Sc. degree of the Andhra University.

have an infinite number of non-negative eigenvalues which can be arranged in non-decreasing order of magnitude. The nth eigenvalue is denoted by λ_n and the associated eigenfunction by ω_n. We further assume that ω_n's form an ortho-normal set. All the eigenvalues are positive for the first boundary value problem, whereas for the second, $\lambda = 0$ is a simple eigenvalue with the eigenfunction $\omega = |D|^{-1/2}$, where $|D|$ is the volume of the bounded domain D. If n is made to run through the values 1, 2, 3, ... in I, and through 0, 1, 2, ... in II, in either case λ_1, λ_2, ... are all positive and $\lambda_0 = 0$. Let us now consider the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\omega_n(P)\,\omega_n(Q)}{\lambda_n^{s+\beta/2+k/4}} J_{\beta+k/2}(R\sqrt{\lambda_n}), \quad s = \sigma + i\,\tau, \tag{1.3}$$

where P and Q are two points in the interior of the domain D, $J_\nu(z)$ is the Bessel function of order ν, and β and R are two real numbers. The series (1.3) is absolutely convergent for all sufficiently large values of σ, say $\sigma > \sigma_0$, and consequently its sum represents an analytic function of s for $\sigma > \sigma_0$. It is the aim of this paper to study the analytic continuation of the sum of the series in (1.3) all over the complex s-plane. We denote this function by $\zeta_k(P, Q, \beta; s)$. If in (1.2) the domain is chosen as

$$D \equiv 0 \leqslant x_1, x_2, \ldots, x_k \leqslant 2\pi,$$

and the boundary condition is periodic, i.e.

$$u(x_1,\ldots,x_{r-1}, 0, x_{r+1},\ldots,x_k) = u(x_1,\ldots,x_{r-1}, 2\pi, x_{r+1},\ldots,x_k),$$

$$r = 1, 2, 3, \ldots, k;$$

then the series in (1.3) reduces to the one studied by S. Bochner and K. Chandrasekharan [2].

It is found that, in general, if R is not equal to the distance between P and Q, $\zeta_k(P, Q, \beta; s)$ is an entire function of s, and if R is equal to the distance between P and Q, $\zeta_k(P, Q, \beta; s)$ is a meromorphic function of s with simple poles on the real axis. Further, we suppose throughout that $k \geqslant 2$, as in the case $k = 1$ the series (1.3) can be studied using the properties of Riemann's

Zeta function and the circular functions. Throughout this paper it will be assumed that the sphere with centre P or Q and radius R is entirely contained in the domain D.

2. Preliminaries. In this section we prove certain lemmas needed in the sequel. We start with a lemma which establishes the dependence of the existence or otherwise of the singular points of the function $f(s)$ represented by the Dirichlet series

$$f(s) = \sum_1^\infty a_n \lambda_n^{-s}, \quad s = \sigma + i\tau, \tag{2.1}$$

on the asymptotic expansion of the function $F(x)$, of the real variable x, defined by

$$F(x) = \sum_1^\infty a_n e^{-\lambda_n x}, \tag{2.2}$$

as $x \to +0$. We next prove two more lemmas on the asymptotic expansion of Gauss's integrals.

DEFINITION I. *A function $f(x)$ is said to admit the series*

$$C_0 x^{a_0} + C_1 x^{a_1} + C_2 x^{a_2} + \dots$$

as asymptotic expansion as $x \to 0$, where

$$a_0 < a_1 < a_2 \dots \to \infty,$$

and $a_0, a_1, \dots$ may be negative or positive, integral or non-integral, if

$$\lim_{x\to 0} x^{-a_p}[f(x) - C_0 x^{a_0} C_1 x^{a_1} \dots C_{p-1} x^{a_{p-1}}] = Cp, \quad p = 0, 1, 2, \dots .$$

We express this by writing

$$f(x) \sim C_0 x^{a_0} + C_1 x^{a_1} + \dots .$$

DEFINITION II. *In what follows a function $f(x)$ will be called an L-function if*

$$\lim_{x\to 0} |x^{-n} f(x)| = 0,$$

for all non-negative integral values of n.

EXAMPLE. We observe that $e^{-p/x}$ for $p > 0$ and $x > 0$ is an L-function.

Let us now define two functions $f(s)$ and $F(s)$ by means of Dirichlet series, as follows :

$$f(s) = \sum_{1}^{\infty} a_n \lambda_n^{-s}, \tag{2.3}$$

$$F(s) = \sum_{1}^{\infty} a_n e^{-\lambda_n s}, \quad s = \sigma + i\tau. \tag{2.4}$$

We now state

LEMMA 2.1. *If the series in* (2.3) *has a domain of convergence, and* $F(x)$ *in* (2.4), *for real* x, *admits the asymptotic expansion given by*

$$F(x) \sim c_0 x^{\alpha_0} + c_1 x^{\alpha_1} + \ldots, \tag{2.5}$$

as $x \to 0$, *then* $f(s)$ *of* (2.3) *is analytic everywhere in the complex* s*-plane except at the points* $s = -\alpha_r$ $(r = 0, 1, 2 \ldots)$ *where it may have simple poles. Also* $f(s)$ *may have simple zeros at* $s = 0, -1, -2, \ldots$.

PROOF. This is a special case of a theorem on the Mellin transform of a Dirichlet series [5, p. 261] and can be proved following the method of the theorems of Hardy and Fekete [1, pp 179-183].

SPECIAL CASES OF LEMMA 2.1. (1). If $\alpha_r = q + r$, $(r = 0, 1, 2, \ldots)$, where q is a positive integer or zero, $f(s)$ is an entire function of s. It has zeros at $s = 0, -1, -2, \ldots, -(q-1)$ for $q \geqslant 1$.

(2). If $\alpha_r = q + r$, $(r = 0, 1, 2, \ldots)$, where q is a positive integer, $f(s)$ is a meromorphic function of s with a finite number of simple poles at $s = q, q-1, \ldots, 1$.

(3). If $F(x)$ is an L-function, $f(s)$ is an entire function of s with zeros at $s = 0, -1, -2, \ldots$.

We indicate the proof of (3) as it will serve as a useful reference later on. When $F(x)$ is an L-function,

$$\int_0^{\xi_1} F(x)\, x^{s-1}\, dx, \quad \xi_1 > 0, \tag{2.6}$$

is finite for all values of s and hence is an entire function of s. Proceeding as in the proof of Lemma (2.1), we can write

$$\frac{1}{\Gamma(s)} \int_0^{\infty} F(x)\, x^{s-1}\, dx = \frac{E(s)}{\Gamma(s)}, \tag{2.7}$$

where $E(s)$ is an entire function, and $\Gamma(s)$ is the Gamma function. From (2.7) the special case (3) follows.

LEMMA 2.2. *Let us define a function* $G(t)$ *as follows* :

$$G(t) = \int_a^b e^{-(y-\alpha)^2/4t} g(y)\, dy,\ t > 0\ . \tag{2.8}$$

If α *lies outside the interval* (a, b), *and* $g(y)$ *is absolutely integrable in* (a, b), *then* $G(t)$ *is an L-function.*

PROOF. For definiteness we assume that $a < b < \alpha$, and have

$$\left| \int_a^b e^{-(y-\alpha)^2/4t} g(y)\, dy \right| < e^{-(b-\alpha)^2/4t} \int_a^b |\, g(y)\,|\, dy. \tag{2.9}$$

Since $g(y)$ is absolutely integrable in (a, b), the right member of (2.9) is an L-function. Therefore $G(t)$ also is an L-function.

LEMMA 2.3. *If* $g(y)$ *is absolutely integrable in* (a, b), *and as* $y \to \alpha$, *we have*

$$g_\alpha(y) \sim b_0\, y^{\alpha_0} + b_1\, y^{\alpha_1} + \ldots, \tag{2.10}$$

where

$$g_\alpha(y) = \begin{cases} \tfrac{1}{2}[g(\alpha + y) + g(\alpha - y)], & a < \alpha < b, \\ g(a + y), & a = \alpha, \\ g(b - y), & b = \alpha; \end{cases} \tag{2.11}$$

then the asymptotic expansion of $G(t)$ *as* $t \to 0$ *is given by*

$$\sum_0^\infty 2^{\alpha_r+1} b_r\, \Gamma\{(\alpha_r + 1)/2\}\, t^{(\alpha_r+1)/2},\quad a < \alpha < b; \tag{2.12}$$

$$\sum_0^\infty 2^{\alpha_r} b_r\, \Gamma\{(\alpha_r + 1)/2\}\, t^{(\alpha_r+1)/2},\ a = \alpha \text{ or } b = \alpha \tag{2.13}$$

PROOF. We shall first consider the case $a < \alpha < b$. We have

$$\int_a^b e^{-(y-\alpha)^2/4t} g(y)\, dy = \int_a^{\alpha-\epsilon} \ldots dy + \int_{\alpha-\epsilon}^{\alpha+\epsilon} \ldots dy + \int_{\alpha+\epsilon}^b \ldots dy,\ \epsilon > 0. \tag{2.14}$$

The fiist and the third integrals on the right of (2.14) are L-functions by Lemma (2.2). With the help of the substitution

$$y - \alpha = 2u\sqrt{t}, \tag{2.15}$$

the second integral can be written as

$$2\sqrt{t}\int_{-\epsilon/2\sqrt{t}}^{\epsilon/2\sqrt{t}} e^{-u^2} g(a+2u\sqrt{t})\,du, \tag{2.16}$$

which is the same as

$$2\sqrt{t}\int_0^{\epsilon/2\sqrt{t}} e^{-u^2}[g(a+2u\sqrt{t})+g(a-2u\sqrt{t})]\,du$$

$$= 4\sqrt{t}\int_0^{\epsilon/2\sqrt{t}} e^{-u^2} g_a(2u\sqrt{t})\,du, \tag{2.17}$$

in view of (2.11). To establish the asymptotic expansion (2.12), it is sufficient to show that the difference between the integral on the right of (2.17) and the first p terms of (2.12) is $O\{t^{(a_p+1)/2}\}$. We now have, using the formula for the Gamma function,

$$4\sqrt{t}\int_0^{\epsilon/2\sqrt{t}} e^{-u^2} g_a(2u\sqrt{t})\,du - \sum_0^{p-1} 2^{a_r+1} b_r \Gamma\{(a_r+1)/2\}\, t^{(a_r+1)/2}$$

$$= 4\sqrt{t}\int_0^{\epsilon/2\sqrt{t}} e^{-u^2}\left[g_a(2u\sqrt{t}) - \sum_0^{p-1} b_r(2u\sqrt{t})^{a_r}\right]du -$$

$$- \sum_0^{p-1} 2(2\sqrt{t})^{a_r+1} b_r \int_{\epsilon/2\sqrt{t}}^{\infty} e^{-u^2} u^{a_r}\,du$$

$$= K_1 - K_2, \text{ say.} \tag{2.18}$$

In view of (2.10) we can write

$$g_a(2u\sqrt{t}) = \sum_0^{p-1} b_r(2u\sqrt{t})^{a_r} + O\{(2u\sqrt{t})^{a_p}\}, \tag{2.19}$$

and therefore we observe that

$$|K_1| = 4\sqrt{t}\int_0^{\epsilon/2\sqrt{t}} e^{-u^2}\,.\,O\{(2u\sqrt{t})^{a_p}\}\,du$$

$$< 2^{a_p+2}\, t^{(a_p+1)/2}\int_0^{\infty} e^{-u^2}\,.\,O(u^{a_p})\,du$$

$$= O(t^{(a_p+1)/2}). \tag{2.20}$$

As for K_2, we have

$$|K_2| < e^{-\epsilon^2/8t} \sum_0^{p-1} 2(2\sqrt{t})^{a_r+1} b_r \int_0^{\infty} e^{-\frac{1}{2}u^2} u^{a_r}\,du$$

$$= e^{-\epsilon^2/8t} P(t) \quad , \quad P(t) = O(t^{(a_0+1)/2}). \tag{2.21}$$

Hence we have

$$K_1 - K_2 = O\left(t^{(a_p+1)/2}\right). \tag{2.22}$$

In the case $a = \alpha$, instead of (2.14) we get

$$\int_a^b e^{-(y-a)^2/4t} g(y)\, dy = \int_a^{a+\epsilon} \dots dy + \int_{a+\epsilon}^b \dots dy, \epsilon > 0. \tag{2.23}$$

The second integral on the right of (2.23) is an L-function by Lemma (2.2). After the substitution (2.15) the first integral becomes

$$2\sqrt{t} \int_0^{\epsilon/2\sqrt{t}} e^{-u^2} g(a + 2u\sqrt{t})\, du = 2\sqrt{t} \int_0^{\epsilon/2\sqrt{}} e^{-u^2} g_a(2u\sqrt{t})\, du, \tag{2.24}$$

from (2.11). Now, proceeding as before, the asymptotic expansion (2.13) can be obtained. Thus the lemma is proved.

COROLLARY. *If $g(y)$ is absolutely integrable, and as $y \to \alpha$, if*

$$g(y) \sim b_0 + b_1 (y - \alpha) + b_2 (y - \alpha)^2 + \dots, \tag{2.25}$$

then

$$G(t) \sim \begin{cases} \sum\limits_0^\infty 2^{2r+1} b_{2r} \Gamma\{\tfrac{1}{2}(2r+1)\}\, t^{(r+1)/2}, & a < \alpha < b; \\ \sum\limits_0^\infty 2^r b_r \Gamma\{\tfrac{1}{2}(r+1)\}\, t^{(r+1)/2}, & a = \alpha; \\ \sum\limits_0^\infty (-1)^r 2^r b_r \Gamma\{\tfrac{1}{2}(r+1)\}\, t^{(r+1)/2}, & b = \alpha. \end{cases} \tag{2.26}$$

3. The function $\zeta_k(P, Q, \beta; s.)$.

In this section we study the behaviour of the function $\zeta_k(P, Q, \beta; s)$ for all real values of β. The case $\beta = -1$ has been studied already [8]. We first deal with the case $\beta > -1$, with the help of the lemmas of the previous section and then extend the results to the case $\beta < -1$ with the help of the recurrence relation for $J_\nu(z)$.

We have already defined $\zeta_k(P, Q, \beta; s)$ with the help of the Dirichlet series

$$\sum_1^\infty \frac{\omega_n(P)\, \omega_n(Q)}{\lambda_n^{s+\beta/2+k/4}} J_{\beta+k/2}(R\sqrt{\lambda_n}). \quad s = \sigma + i\tau, \tag{3.1}$$

which, for sufficiently large values of σ, can be obtained as the Mellin transform of the sum of the series

$$\sum_1^\infty \frac{\omega_n(P)\, \omega_n(Q)\, J_{\beta+k/2}}{\lambda_n^{\beta/2+k/4}} (R\sqrt{\lambda_n})\, e^{-\lambda_n t}, \tag{3.2}$$

which is absolutely convergent for $t > 0$. In fact if $F(t)$ denotes the sum of the series in (3.2), then for sufficiently large values of σ,

$$\zeta_k(P, Q, \beta; s) = \frac{1}{\Gamma(s)} \int_0^\infty F(t)\, t^{s-1}\, dt. \tag{3.3}$$

In view of Lemma (2.1) we can determine the nature of the function $\zeta_k(P, Q, \beta; s)$ if we obtain the asymptotic expansion of $F(t)$ as $t \to +0$. We therefore proceed to discuss the asymptotic behaviour of $F(t)$. We start from the following formula (13) of [8].

$$\sum_{n=1}^{\infty} \frac{\omega_n(P)\, \omega_n(Q)\, e^{-\lambda_n t}}{(\rho\sqrt{\lambda_n})^{k/2-1}} J_{k/2-1}(\rho\sqrt{\lambda_n})$$
$$= \frac{1}{(2\sqrt{\pi t})^k} \frac{e^{-(\rho^2+d^2)/4t}\, I_{k/2-1}(\rho d/2t)}{(\rho d/2t)^{k/2-1}} + \tag{3.4}$$
$$+ \frac{2^{1-k/2}}{\Omega\, \Gamma(k/2)} \left[\int\int_\Omega g(Z, Q; t)\, d\Omega - \frac{\delta\Omega}{|D|} \right],$$

wherein R is replaced by ρ, the summation symbol is denoted by n and d represents the distance between P and Q. If we now multiply both sides of (3.4) by $\rho^{k-1}(R^2 - \rho^2)^\beta$, integrate with respect to ρ from 0 to R, change the order of summation and integration on the left, (which is justifiable on account of the absolute convergence of the series for $t > 0$), and use formula (1) of [9, p. 373], we observe that for $\beta > -1$, $k > 0$, and $t > 0$,

$$F(t) = \frac{2^{-(\beta+k/2+1)}\, t^{-1}}{R^{\beta+k/2}\, \Gamma(\beta+1)\, \pi^{k/2}\, d^{k/2-1}} \times$$
$$\times \int_0^R e^{-(\rho^2+d^2)/4t} \rho^{k/2} (R^2-\rho^2)^\beta I_{k/2-1}\left(\frac{\rho d}{2t}\right) d\rho +$$
$$+ \frac{2^{1-\beta-k/2}}{R^{\beta+k/2}\Gamma(\beta+1)\Gamma(k/2)\,\Omega} \int_0^R \int_\Omega g(z, Q; t)\, \rho^{k-1}(R^2-\rho^2)^\beta\, d\Omega\, d\rho -$$
$$- \frac{\delta\, R^{\beta+k/2}}{2^{\beta+k/2}\, |D|\, \Gamma(\beta+k/2+1)},$$
$$= K_1' + K_2' + K_3', \text{ say.} \tag{3.5}$$

K_3' is a constant which becomes zero if $\delta = 0$ and K_2' is an L-function because, by formula (6) of [8],

$$|K_2'| < c'\, t^{-k/2}\, e^{-l_Q^2/4t}, \tag{3.6}$$

where c' is some constant and $l_Q \neq 0$. Therefore the asymptotic expansion of $F(t)$ is the same as that of K_1' except for the constant K_3'. We have to consider the various cases, k-odd and k-even and $R \lesseqgtr d$, and we state the results in the following lemmas.

k-ODD, $\beta > -1$. In this case, from formula (10), [9, p. 80],

$$K_1' = M \sum_0^{(k-3)/2} \frac{(-1)^r \{(k-3)/2 + r\}!\, t^{r-1/2}}{r!\{(k-3)/2 - r\}!\, d^r} \times$$

$$\times \int_0^R e^{-(\rho-d)^2/4t} \rho^{(k-1-2r)/2} (R^2 - \rho^2)^\beta \, d\rho +$$

$$+ (-1)^{(k-1)/2} M \sum_0^{(k-3)/2} \frac{\{(k-3)/2 + r\}!\, t^{r-1/2}}{r!\{(k-3)/2 - r\}!\, d^r} \times$$

$$\times \int_0^R e^{-(\rho+d)^2/4t} \rho^{(k-1-2r)/2} (R^2 - \rho^2)^\beta \, d\rho,$$

$$= S_1 - S_2, \text{ say,} \tag{3.7}$$

where

$$M = [\, 2^{-(1+\beta+k/2)} / \Gamma(\beta+1)\, R^{\beta+k/2} \pi^{(k+1)/2} d^{(k-1)/2} \,]. \tag{3.8}$$

We at once observe that S_2 is an L-function because it is the sum of a finite number of terms each of which is an L-function by Lemma (2.2) so that K_1' has the same asymptotic expansion as that of S_1.

LEMMA 3.1. *If $R < d$, $F(t)$ is K_3' + an L-function.*

PROOF. Since $F(t)$ has the same asymptotic expansion as K_1' (apart from the constant K_3'), which in its turn has the same asymptotic expansion as S_1, we have only to prove that S_1 is an L-function. This is so, since S_1 also now is an L-function like S_2.

LEMMA 3.2. *If $R > d$, then*

$$F(t) \sim \sum_0^p A_n t^n,$$

where

$$p = \begin{cases} \beta + (k-3)/2, & \beta \text{ is a non-negative integer,} \\ \infty & , \beta \text{ is not a non-negative integer:} \end{cases}$$

Proof. As in Lemma (3.1) we have to discuss only S_1. Let us consider the general term of S_1, namely

$$\frac{(-1)^r M\{(k-3)/2+r\}!\, t^{r-1/2}}{r!\,\{(k-3)/2-r\}!\, d^r} \times$$

$$\times \int_0^R e^{-(\rho-d)^2/4t} \rho^{(k-1-2r)/2} (R^2-\rho^2)^\beta \, d\rho. \tag{3.9}$$

If we set

$$g^{1,r}(\rho) = \rho^{(k-1-2r)/2} (R^2-\rho^2)^\beta, \tag{3.10}$$

we observe that

$$g^{1,r}(\rho) = \sum_0 b^{1,r}_n (\rho-d)^n, \tag{3.11}$$

where the right side of (3.11) is the Taylor expansion of $g^{1,r}(\rho)$ in the neighbourhood of $\rho = d$. It contains a finite number of terms if β is a non-negative integer, for, k being odd, $(k-1-2r)/2$ is a positive integer for $0 \leqslant r \leqslant (k-3)/2$, and then $g^{1,r}(\rho)$ is a polynomial in ρ. If β is not a non-negative integer the Taylor expansion contains an infinite number of terms. Applying the corollary to Lemma (2.3) the asymptotic expansion of (3.9) can be written as

$$\sum_0 c^{1,r}_n t^{n+r}, \tag{3.12}$$

where $c^{1,r}_n$'s are related to $b^{1,r}_n$'s (Lemma 2.3). Writing down the asymptotic expansion of each of the terms in S_1 similarly, and adding them up we obtain the asymptotic expansion of S_1 as

$$\sum_0^p A_n t^n, \tag{3.13}$$

where A_n's can be computed from $c^{1,r}_n$'s and p may be finite or infinite according as β is a non-negative integer or not. K_3' is absorbed in A_0. In fact

$$A_0 = 2\,\Gamma(1/2)\, M\, (R^2-d^2)^\beta d^{(k-1)/2} + K_3'. \tag{3.14}$$

The value of p when β is a non-negative integer can be determined thus. From (3.11) and the corollary to Lemma 2.3 the highest power of t in the asymptotic expansion of (3.9) is

$$\beta + r/2 + (k-1)/4 \text{ or } \beta + r/2 + (k-3)/4, \tag{3.15}$$

whichever is an integer, and this is greatest for $r = (k-3)/2$. Hence $p = \beta + (k-3)/2$. This proves the lemma.

LEMMA 3.3. *If $R = d$, then*

$$F(t) \sim t^{\beta/2} \sum_{0}^{p} A_n' \, t^{n/2} + K_3',$$

where

$$p = \begin{cases} \beta + k - 2\,, & \beta \text{ is a non-negative integer,} \\ \infty \,, & \beta \text{ is not a non-negative integer.} \end{cases}$$

PROOF. As in the proof of Lemma (3.2) we consider the general term of S_1 given by (3.9), and set

$$g^{2,r}(\rho) = \rho^{(k-1-2r)/2} (d^2 - \rho^2)^\beta . \tag{3.16}$$

In accordance with (2.11) we write

$$g_d^{2,r}(\rho) = g^{2,r}(d-\rho) = (2\,d\rho)^\beta (d-\rho)^{(k-1-2r)/2} (1 - \rho/2d)^\beta$$
$$= \sum_{0}^{p} b_n^{2,r} \rho^{\beta+n}\,, \tag{3.17}$$

where the sum in (3.17) may contain a finite or an infinite number of terms according as β is a non-negative integer or not, as explained in Lemma (3.2). Now applying (2.13), the asymptotic expansion of (3.9) can be written as

$$\sum_{n=0} c_n^{2,r} \, t^{(\beta+n)/2+r}. \tag{3.18}$$

Writing down the asymptotic expansion of all the terms of S_1 in a similar way, and adding them all, we get

$$S_1 \sim t^{\beta/2} \sum_{0}^{p} A_n' \, t^{n/2}. \tag{3.19}$$

In particular

$$A_0' = (4\,d)^\beta M \, d^{(k-1)/2} \, \Gamma\{(\beta+1)/2\}.$$

When β is a non-negative integer, p is determined thus. From (2.13) and (3.17) the highest value of n in (3.18) is $\beta + (k-1-2r)/2$. Therefore the highest power of t in (3.18) is $\beta + (k-1+2r)/2$. The highest power of t in (3.19) is attained for $r = (k-3)/2$. Hence $p = \beta + k - 2$. We now state the theorems.

THEOREM 3.1. *Let P and Q be two distinct points in the interior of the domain D, d the distance between them and R be such that*

the sphere with centre P or Q and radius R is completely contained within the domain D. Let k be odd and $\beta > -1$. Then if $R < d$, $\zeta_k(P, Q, \beta; s)$ is an entire function of s with zeros at $s = 0, -1, \ldots$, or at $s = -1, -2, \ldots$, according as the boundary condition I *or* II *is assumed.*

Proof. From Lemma 3.1 the asymptotic expansion of $F(t)$ is $K_3' +$ an L-function, and K_3' is zero or not according as $\delta = 0$ or not, i.e. according as the boundary condition I or II is assumed. Hence from Lemma 2.1, special case (3), the result follows.

Theorem 3.2. *With the same hypothesis as in Theorem* 3.1, *if $R > d$, $\zeta_k(P, Q, \beta; s)$ is an entire function of s. Further if β is an integer $\zeta_k(P, Q, \beta; s)$ has zeros at $s = -(\beta + \overline{k-1}/2), -(\beta + \overline{k+1}/2), \ldots$*

Proof. From Lemma 3.2 and special case (1) of Lemma 2.1 it at once follows that $\zeta_k(P, Q, \beta; s)$ is an entire function of s. If β is a non-negative integer it is easily seen from Lemmas 2.1 and 3.2 that the zeros at $s = -r\,(r = \beta + (k-1)/2, \beta + (k+1)/2, \ldots)$ remain.

Theorem 3.3. *With the same hypothesis as in Theorem* 3.1, *if $R = d$, $\zeta_k(P, Q, \beta; s)$ is a meromorphic function of s with*

(*i*) *a finite number of simple poles at $s = -\beta/2, -(\beta + 2)/2, \ldots, -(\beta + \overline{k-2}/2)$, and zeros at $s = 0, -1, -2, \ldots, -(\overline{\beta - 1}/2)$, $-(\beta + \overline{k-1}/2), -(\beta + \overline{k+1}/2), \ldots$, if β is odd,*

(*ii*) *a finite number of simple poles at $s = -(\beta+1)/2, -(\beta+3)/2 \ldots$, $-(\beta + \overline{k-2}/2)$, and zeros at $s = 0, -1, \ldots, -(\beta - 2)/2$, $-(\beta + \overline{k-1}/2), -(\beta + \overline{k+1}/2), \ldots$, if β is even,*

(*iii*) *an infinite number of simple poles at $s = -\beta/2, -(\beta + 1)/2$, $(\beta + 2)/2, \ldots$, and zeros at $s = 0, -1, -2, \ldots$, if β is non-integral. The zero at $s = 0$ does not appear in the case of the boundary condition* II.

Proof. From Lemmas 3.3 and 2.1, $\zeta_k(P, Q, \beta; s)$ has poles at $s = 0, -\beta/2, -(\beta+1)/2, -(\beta+2)/2, \ldots, -(\beta + \overline{k-2}/2)$,

if $\delta \neq 0$. The pole at $s = 0$ disappears if $\delta = 0$. It has zeros at $s = 0, -1, \ldots$. If $\delta \neq 0$ the pole at $s = 0$ and the zero at $s = 0$ cancel each other. If $\delta = 0$ the zero at $s = 0$ remains. Now, if β is an odd integer the poles at $s = -\overline{(\beta + n/2)}$, $(n = 1, 3, 5, \ldots, \beta + k - 3)$ are cancelled by the zeros at the corresponding points. The poles and zeros remaining agree with those mentioned in the theorem. If β is an even integer the poles at $s = -\overline{(\beta + n/2)}$, $(n = 0, 2, \ldots, \beta + k - 3)$ are cancelled by the zeros at the corresponding points. The remaining poles and zeros agree with those given by the theorem. The result for β non-integral also follows from the above considerations.

k-EVEN, $\beta > -1$. In this case we have to make use of the formula (2) [9, p. 203], which is true for large values of $(\rho d/2t)$, that is, for small values of t for fixed ρ and d different from zero. We write

$$K_1' = M t^{-1} \sqrt{(\pi d)} \int_0^{\mu t/d} e^{-(\rho^2+d^2)/4t} \rho^{k/2} (R^2 - \rho^2)^\beta I_{k/2-1}\left(\frac{\rho d}{2t}\right) d\rho + \\ + M t^{-1} \sqrt{(\pi d)} \int_{\mu t/d}^{R} \ldots d\rho, \tag{3.20}$$

where M is defined in (3.8) and $\mu > 0$, $0 < \mu t/d < R$. Employing formula (2) [9, p. 77], the first integral can be expressed as

$$M t^{-1} \sqrt{(\pi d)} \int_0^{\mu t/d} e^{-(\rho^2+d^2)/4t} \rho^{k/2} (R^2 - \rho^2)^\beta \left\{ \sum_0^\infty \frac{(\rho d/4t)^{2r+k/2-1}}{r!\,\Gamma(r + k/2)} \right\} d\rho, \tag{3.21}$$

which is absolutely less than

$$M t^{-1} \sqrt{(\pi d)}\, e^{-d^2/4t} \sum_0^\infty \frac{(\mu/4)^{2r+k/2-1}}{r!\,\Gamma(r + k/2)} \int_0^{\mu t/d} \rho^{k/2} (R^2 - \rho^2)^\beta \, d\rho$$

$$< M t^{-1} \sqrt{(\pi d)}\, e^{-d^2/4t} \sum_0^\infty \frac{(\mu/4)^{2r+k/2-1}}{r!\,\Gamma(r + k/2)} \int_0^{R} \rho^{k/2} (R^2 - \rho^2)^\beta d\rho. \tag{3.22}$$

Since the series in (3.22) is convergent for a finite μ, however large it may be, and the integral on the right of (3.22) is convergent for $k > 0, \beta > -1$, it follows that (3.21) is an L-function. Therefore

K_1' has the same asymptotic expansion as the second integral on the right of (3.20). This second integral can be written, in view of formula (2) [9, p. 203], as

$$M \int_{\mu t/d}^{R} e^{-(\rho-d)^2/4t} \rho^{(k-1)/2} (R^2 - \rho^2)^\beta \times$$

$$\times \left[\sum_{0}^{p-1} \frac{(-1)^r (k/2-1, r)\, t^{r-1/2}}{(\rho d)^r} + E_p\left(\frac{\rho d}{2t}\right) \frac{t^{p-1/2}}{(\rho d)^p} \right] d\rho +$$

$$+ M e^{\frac{1}{2}(k-1)\pi i}\, t^{-1/2} \int_{\mu t/d}^{R} e^{-(\rho+d)^2/4t} \rho^{(k-1)/2} (R^2 - \rho^2)^\beta \left[1 + E_0\left(\frac{\rho d}{2t}\right)\right] d\rho,$$

$$= T_1 + T_2, \text{ say,} \tag{3.23}$$

where $E_p(\rho d/2t)$ and $E_0(\rho d/2t) = o(1)$ as $t \to +0$, and p is an arbitrary positive integer. We now have

$$|T_2| < M t^{-1/2} \int_0^R e^{-(\rho+d)^2/4t} \rho^{(k-1)/2} (R^2 - \rho^2)^\beta \left[1 + E_0\left(\frac{\rho d}{2t}\right)\right] d\rho. \tag{3.24}$$

From (3.24) and Lemma 2.2 it follows that T_2 is an L-function. Thus K_1' has the same asymptotic expansion as T_1 which has to be discussed under various cases $R \lesseqgtr d$. We proceed to do so in the following lemmas.

LEMMA 3.4. *If $R < d$, $F(t)$ is K_3' + an L-function.*

PROOF. We have already shown that $F(t)$ has the same asymptotic expansion as K_1', (but for the constant K_3'), and K_1' is just now observed to have the same asymptotic expansion as T_1. The lemma is proved therefore if we show that T_1 is an L-function. In this case since $R < d$, putting $p = 0$, T_1 can be dealt with like T_2 and shown to be an L-function.

LEMMA 3.5. *If $R > d$, then*

$$F(t) \sim \sum_{0}^{\infty} B_n t^n.$$

PROOF. As in Lemma 3.4 we have to consider the asymptotic expansion of T_1 only. We split up the interval of integration $(\mu t/d, R)$ into $(\mu t/d, d-\epsilon)$ and $(d-\epsilon, R)$, where $\epsilon > 0$, is a constant.

The first interval does not contain the point $\rho = d$, and the second is free from the variable point $\rho = \mu t/d$. We now write

$$T_1 = \int_{\mu t/d}^{d-\epsilon} \dots d\rho + \int_{d-\epsilon}^{R} \dots d\rho. \tag{3.25}$$

The first integral is one like T_2, and can be shown to be an L-function. The second is

$$\sum_{0}^{p-1} \frac{(-1)^r M(k/2-1, r)\, t^{r-1/2}}{d^r} \int_{d-\epsilon}^{R} e^{-(\rho-d)^2/4t} \rho^{(k-1-2r)/2} (R^2-\rho^2)^\beta \, d\rho +$$

$$+ \frac{Mt^{p-1/2}}{d^p} \int_{d-\epsilon}^{R} e^{-(\rho-d)^2/4t} E_p\left(\frac{\rho d}{2t}\right) \rho^{(k-1-p)/2} (R^2-\rho^2)^\beta d\rho,$$

$$= S_3 + T_3, \text{ say,} \tag{3.26}$$

where $E_p(\rho d/2t) = o(1)$ as $t \to +0$. By considering the asymptotic expansion of T_3, with the help of the corollary to Lemma 2.3 it can be shown that $T_3 = o(t^p)$ as $t \to +0$.

Proceeding as in Lemma 3.2 the asymptotic expansion of S_3 can be evaluated. We observe that, since $(k-1-2r)/2$ is no longer an integer, the asymptotic expansion of S_3 contains an infinite number of terms irrespective of β being an integer or not. We write

$$S_3 \sim \sum_{0}^{p-1} B_n t^n + O(t^p), \tag{3.27}$$

where B_n is independent of p. This shows that the first p terms of the asymptotic expansions of $F(t)$ and S_3 are the same. Hence

$$F(t) \sim \sum_{0}^{p-1} B_n t^n + O(t^p). \tag{3.28}$$

It is to be observed that the first p terms in (3.28) do not depend on p, since even if we take more terms in (3.26) they do not make any contribution to the first p terms. Since p can be made as large as we like, the lemma follows.

LEMMA 3.6. *If $R = d$, then*

$$F(t) \sim t^{\beta/2} \sum_{0}^{\infty} B_n' t^{n/2} + K_3'.$$

PROOF. This can be discussed as we did Lemma 3.5. Applying the result (2.13) this Lemma can be proved like Lemma 3.3. We now state

THEOREM 3.4. *Let P and Q be two distinct points in the interior of the domain D, d be the distance between them, and R be such that a sphere with centre P or Q and radius R is completely contained in the domain D. Let k be even and* $\beta > -1$. *Then if* $R < d$, $\zeta_k(P, Q, \beta; s)$ *is an entire function of s with zeros at* $s = 0, -1, -2, \ldots$ *or at* $s = -1, -2, \ldots$ *according as boundary condition* I *or* II *is assumed.*

PROOF. This theorem follows from Lemma 3.4 in a manner similar to that of Theorem 3.1.

THEOREM 3.5. *With the same hypothesis as in Theorem* 3.4 *if* $R > d$, $\zeta_k(P, Q, \beta; s)$ *is an entire function of s.*

PROOF. This follows from Lemma 3.5 and the special case (1) of Lemma 2.1.

THEOREM 3.6. *With the same hypothesis as in Theorem* 3.4 *if* $R = d$, $\zeta_k(P, Q, \beta; s)$ *is a meromorphic function of s with*

(i) *an infinite number of simple poles at* $s = -\beta/2, -(\beta+2)/2, \ldots,$ *and zeros at* $s = 0, -1, -2, \ldots, -(\beta-1)/2$, *if β is an odd integer,*

(ii) *an infinite number of simple poles at* $s = -(\beta+1)/2, -(\beta+3)/2, \ldots,$ *and zeros at* $s = 0, -1, -2, \ldots, -(\beta-2)/2$, *if β is an even integer,*

(iii) *with an infinite number of poles at* $s = -\beta/2, -(\beta+1)/2, -(\beta+2)/2, \ldots,$ *and zeros at* $s = 0, -1, -2, \ldots,$ *if β is not an integer. The zero at* $s = 0$ *does not appear in the case of boundary condition* II.

PROOF. This theorem can be proved like Theorem 3.3 with the help of the Lemma 3.6.

P AND *Q* COINCIDENT. We shall introduce the function $\zeta_k(P, \beta; s)$ as the analytic extension of the sum of the series

$$\sum_{1}^{\infty} \frac{\omega_n^2(P)\, J_{\beta+k/2}(R\sqrt{\lambda_n})}{\lambda_n^{\beta/2+k/4}}, \quad s = \sigma + i\tau. \tag{3.29}$$

It is observed that in (3.1) if we put $P = Q$ we obtain (3.29). We have corresponding to (3.2)

$$\sum_{1}^{\infty} \frac{\omega_n^2(P)\, J_{\beta+k/2}(R\sqrt{\lambda_n})\, e^{-\lambda_n t}}{\lambda_n^{\beta/2+k/4}}. \tag{3.30}$$

We require the asymptotic expansion of (3.30) to determine the behaviour of $\zeta_k\,(P,\,\beta;\,s)$. Once again borrowing the equation (29) of [8], we have

$$\sum_1^\infty \frac{\omega_n^2(P)\,\Gamma(k/2)\,J_{k/2-1}(\rho\sqrt{\lambda_n})\,e^{-\lambda_n t}}{\{(\rho\sqrt{\lambda_n})/2\}^{k/2-1}} = \frac{e^{-\rho^2/4t}}{(2\sqrt{(\pi t)})^k} + \\ + \frac{1}{\Omega}\iint_\Omega g(x,\,P;\,t)\,d\Omega - \frac{\delta}{|D|}\,. \qquad (3.31)$$

Multiplying both sides by $\rho^{k-1}\,(R^2-\rho^2)^\beta$ and integrating with respect to ρ from 0 to R, we get for $\beta > -1$, $k > 0$, and $t > 0$, from the formula (1) [9, p. 373]

$$F(t) = \sum_1^\infty \frac{\omega_n{}^2(P)\,J_{\beta+k/2}\,(R\sqrt{\lambda_n})\,e^{-\lambda_n t}}{\lambda_n{}^{\beta/2+k/4}}$$

$$= c_1\int_0^R e^{-\rho^2/4t}\,\rho^{k-1}\,(R^2-\rho^2)^\beta\,d\rho +$$

$$+ c_2\int_0^R\int_\Omega g(x,\,P\,;\,t)\;\rho^{k-1}\,(R^2-\rho^2)^\beta\;d\Omega\,d\rho +$$

$$+ c_3\int_0^R \rho^{k-1}\,(R^2-\rho^2)^\beta\,d\rho, \qquad (3.32)$$

where the constants c_1, c_2, c_3 are easily obtained from (3.31). On the right of (3.32) we note that the second integral is already shown to be an L-function and the third is a constant. The asymptotic expansion of the first can be written down with the help of (2.13) in the form

$$\sum_0^p B^2{}_n\,t^n, \qquad (3.33)$$

where p is finite or infinite according as β is a non-negative integer or not. When β is a non-negative integer it can be easily shown that $p = \beta$. Hence we have

$$F(t) \sim \sum_0^p B^2{}_n\,t^n + \text{constant}. \qquad (3.34)$$

We now state

THEOREM 3.7. *Let P be an interior point of the domain D and R be such that a sphere of radius R and centre P is completely*

contained in D. Then for $\beta > -1$, $\zeta_k(P, \beta; s)$ is an entire function of s. If β is a non-negative integer $\zeta_k(P, \beta; s)$ has zeros at $s = -(\beta + 1), -(\beta + 2), \ldots$.

PROOF. This follows from (3.34) with the help of special case 1, Lemma 2.1.

$\zeta_k(P, Q, \beta; s)$ FOR ANY REAL β. We shall now remove the restriction $\beta > -1$. The case $\beta = -1$ has been already studied [8]. If $\beta < -1$ the formula (1) [9, p. 373] can no longer be used and therefore we proceed to obtain a formula making use of the recurrence relation between Bessel functions. By means of this formula we shall make the discussion of the case $\beta < -1$ depend upon the results in the cases $\beta \geqslant -1$. We have from formula (1) [9, p. 45].

$$\sum_1^\infty \frac{\omega_n(P)\,\omega_n(Q)}{\lambda_n^{s+\beta/2+k/4}} J_{\beta+k/2}(R\sqrt{\lambda_n})$$

$$= \frac{2\beta+k+2}{R}\sum_1^\infty \frac{\omega_n(P)\,\omega_n(Q)}{\lambda_n^{s+(\beta+1)/2+k/4}} J_{\beta+1+k/2}(R\sqrt{\lambda_n}) -$$

$$-\sum_1^\infty \frac{\omega_n(P)\,\omega_n(Q)}{\lambda_n^{s-1+(\beta+2)/2+k/4}} J_{\beta+2+k/2}(R\sqrt{\lambda_n}), \tag{3.35}$$

which can be written as

$$\zeta_k(P, Q, \beta; s) = \frac{2\beta+k+2}{R}\zeta_k(P, Q, \beta+1; s) - \zeta_k(P, Q, \beta+2; s-1). \tag{3.36}$$

If β is such that $-2 \leqslant \beta < -1$, by using (3.36) the nature of $\zeta_k(P, Q, \beta; s)$ can be deduced from that of $\zeta_k(P, Q, \beta+1; s)$ and $\zeta_k(P, Q, \beta+2; s-1)$ which are already known since $\beta + 1 \geqslant -1$. In fact, a repeated application of the above formula will enable us to deduce for $-(p+1) \leqslant \beta < -1$, where p is the least integer satisfying the inequality, the formula

$$\zeta_k(P,Q,\beta;s) = c_0\zeta_k(P,Q,\beta+p;s) + c_1\zeta_k(P,Q,\beta+p+1;s-1) + + c_2\zeta_k(P,Q,\beta+p;s-1) + c_3\zeta_k(P,Q,\beta+p+1;s-2) + ,\ldots \tag{3.37}$$

where c_0, c_1, c_2 ... are some constants which can be determined. The last term in the above formula is given by

$$\begin{aligned} &- \zeta_k (P, Q, \beta + p; s - p/2), \quad p \text{ even} \\ &- \zeta_k (P, Q, \beta + p + 1; s - \overline{p + 1}/2), \quad p \text{ odd.} \end{aligned} \tag{3.38}$$

In view of the above formula we have the following theorems.

THEOREM 3.8. *If P and Q are two distinct interior points of the domain D, d the distance between them and R is such that the sphere of radius R and centre P or Q is completely contained in the domain D then if $R \neq d$, $\zeta_k (P, Q, \beta; s)$ is an entire function of s for all real values of β.*

THEOREM 3.9. *With the same hypothesis as in Theorem (3.8) if $R = d$, $\zeta_k (P, Q, \beta; s)$ is, for all real values of β, a meromorphic function with simple poles on the real axis. The first pole is at $s = -\beta/2$, if β is not an even integer and $s = -(\beta + 1)/2$ if β is an even integer. The remaining poles are situated to the left of the first pole. When k is odd and β is an integer $\zeta_k (P, Q, \beta; s)$ has a finite number of poles.*

The author has great pleasure in acknowledging his indebtedness to Professor S. Minakshisundaram for his guidance in the preparation of this paper.

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Andhra University
Waltair

ON PAIRS OF TELEPARALLELISMS*

By R. N. SEN

[Received August 26, 1952]

1. Attempts have been made in this paper first to obtain conditions for which a given teleparallelism in a metric space may generate others and secondly to apply these conditions in a 3-dimensional space of a certain kind in order to obtain types of spaces which are compatible with pairs of teleparallelisms so generated.

Let us consider an n-dimensional space with the hypotheses that its metric is given by a quadratic differential form

$$ds^2 = g_{ij}\, dx^i\, dx^j, \tag{1.1}$$

as in Riemannian geometry and that it admits parallel displacements of vectors given by

$$dV^t + \Gamma^t_{ij}\, V^i\, dx^j = 0, \tag{1.2}$$

as in geometry of affine connections. In what follows the covariant derivative of any tensor with respect to (1.2) will be denoted by a comma and the summation convention with respect to indices, when they occur once above and once below, will be followed but other Σ's will be retained, unless otherwise stated.

The following facts are known [2]. If we put

$$a = \Gamma^t_{ij}, \quad a^* = \Gamma^t_{ij} + g^{mt}\, g_{im,j}, \quad a' = \Gamma^t_{ji},$$

then for every parallelism corresponding to a, there exist uniquely the two parallelisms corresponding to a^* and a', which are respectively called the associate and the conjugate of a having the property

$$a^{**} = a'' = a.$$

Specially, a is self-associate if $a = a^*$ and self-conjugate if $a = a'$. Further, if we put

$$\alpha = g^{mt}\, g_{im,j},\ \alpha_c = g^{mt}\, g_{jm,i},\ \gamma = g^{mt}\, g_{ij,m} = \gamma_c,$$
$$\beta = g^{mt}\, g_{is}(\Gamma^s_{mj} - \Gamma^s_{jm}),\ \beta_c = g^{mt}\, g_{js}(\Gamma^s_{mi} - \Gamma^s_{im}),$$

and construct the sequence

$$a_1 = a,\ a_2 = a^*,\ a_3 = a^{*\prime},\ a_4 = a^{*\prime *},\ a_5 = a^{*\prime *\prime},\ldots,$$

then the sequence is a finite cyclic sequence of 12 terms (assuming all distinct) having the following values :

$$\begin{array}{ll} a_1 = a, & a_2 = a + \alpha, \\ a_3 = a' + \alpha_c, & a_4 = a + \alpha + \beta - \gamma, \\ a_5 = a' + \alpha_c + \beta_c - \gamma, & a_6 = a + \alpha + \alpha_c + \beta + \beta_c - \gamma, \\ a_7 = a' + \alpha + \alpha_c + \beta + \beta_c - \gamma, & a_8 = a' + \alpha_c + \beta + \beta_c - \gamma, \\ a_9 = a + \alpha + \beta + \beta_c - \gamma, & a_{10} = a' + \alpha_c + \beta_c, \\ a_{11} = a + \alpha + \beta, & a_{12} = a'. \end{array}$$

Also, the Christoffel symbol $\left\{ {t \atop ij} \right\} = \frac{1}{2}(a_i + a_{i+6}),\ i = 1,\ 2, \ldots.$ (1.3)

2. Let us now make the following specialization. Suppose first that the generating element a is made self-associate by imposing the condition $\alpha = 0$ and therefore $\alpha_c = \gamma = 0$. Then the terms of the above sequence reduce to the following six (assuming all distinct) :

$$\begin{aligned} &a_1 = a_2 = a, \\ &a_3 = a_{12} = a',\ a_4 = a_{11} = a + \beta, \\ &a_5 = a_{10} = a' + \beta_c,\ a_6 = a_9 = a + \beta + \beta_c, \\ &a_7 = a_8 = a' + \beta + \beta_c = b, \text{ say.} \end{aligned} \tag{2.1}$$

Thus a self-associate a makes b also self-associate. Next suppose that the condition $a' = a + \beta$ is further imposed so as to make the remaining elements also self-associate. As $a' = a + \beta$ is equivalent to $a = a' + \beta_c$, this condition makes $b = a'$. We are therefore left with the following two self-associate elements :

$$\begin{aligned} &a_1 = a_2 = a_5 = a_6 = a_9 = a_{10} = a, \\ &a_3 = a_4 = a_7 = a_8 = a_{11} = a_{12} = a' = c, \text{ say.} \end{aligned} \tag{2.2}$$

It is to be noticed that the second condition modifies a further and makes $b = c$. Also, it follows from (1.3) that, for (2.1) and (2.2), we have respectively

$$\left\{ {t \atop ij} \right\} = \tfrac{1}{2}(a + b) \text{ and } \left\{ {t \atop ij} \right\} = \tfrac{1}{2}(a + c). \tag{2.3}$$

The relations (2.1), (2.2) and (2.3) remain true if specially a, b, c can all define teleparallelisms. To make them so, we introduce three orthogonal ennuples ${}^p h_{1|i}, {}^p h_{2|i}, {}^p h_{3|i}$ into the space (1.1), satisfying the usual equations

$$g_{ij} = \sum_p {}^p h_{\lambda|i}\, {}^p h_{\lambda|j}, \quad {}_p h^i_{\lambda|} = g^{ij}\, {}^p h_{\lambda|j}, \quad \lambda = 1, 2, 3,$$

and define the parallelisms corresponding to a, b, c by setting

$$a = {}_p h^t_{1|} \frac{\partial\, {}^p h_{1|i}}{\partial x^j}, \quad b = {}_p h^t_{2|} \frac{\partial\, {}^p h_{2|i}}{\partial x^j}, \quad c = {}_p h^t_{3|} \frac{\partial\, {}^p h_{3|i}}{\partial x^j}. \tag{2.4}$$

From (2.1) and (2.4) we find that

$$\begin{aligned} {}_p h^t_{2|} \frac{\partial\, {}^p h_{2|i}}{\partial x^j} = {}_p h^t_{1|} \frac{\partial\, {}^p h_{1|j}}{\partial x^i} + \sum_p g^{mt} \Bigg\{ {}^p h_{1|i} \left(\frac{\partial\, {}^p h_{1|m}}{\partial x^j} - \frac{\partial\, {}^p h_{1|j}}{\partial x^m} \right) + \\ + {}^p h_{1|j} \left(\frac{\partial\, {}^p h_{1|m}}{\partial x^i} - \frac{\partial\, {}^p h_{1|i}}{\partial x^m} \right) \Bigg\}, \end{aligned}$$

or

$$\sum_p \left({}^p h_{1|i} \frac{\partial\, {}^p h_{1|j}}{\partial x^k} - {}^p h_{2|j} \frac{\partial\, {}^p h_{2|i}}{\partial x^k} \right) = \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}; \tag{2.5}$$

and from (2.2) and (2.4) we obtain

$$ {}^p h^t_{3|} \frac{\partial\, {}^p h_{3|i}}{\partial x^j} = {}^p h^t_{1|} \frac{\partial\, {}^p h_{1|j}}{\partial x^i}, $$

or

$$\sum_p \left({}^p h_{1|i} \frac{\partial\, {}^p h_{1|j}}{\partial x^k} - {}^p h_{3|i} \frac{\partial\, {}^p h_{3|k}}{\partial x^j} \right) = 0. \tag{2.6}$$

The equations (2.5) and (2.6) are the required conditions that a given teleparallelism corresponding to a may generate successively two others corresponding to b and c. That is to say that if the pair of teleparallelisms corresponding to a, b exist satisfying (2.1), namely $b = a' + \beta + \beta_c$, then the ennuples defining them by (2.4) must satisfy the condition (2.5); similarly for the pair a, c. It should however be clearly understood that if the space admits (2.6), it will also admit (2.5) when ${}^p h_{3|i} = {}^p h_{2|i}$, in view of the imposition of the condition $a' = a + \beta$ on (2.5).

3. The above conditions may be put in another form in which a geometrical interpretation can be given. Let the covariant deriva-

tive of tensors with respect to the Levi-Civita parallelism be denoted by curly brackets. It is then known that Ricci's coefficients of rotation with respect to an orthogonal ennuple ${}^{p}h_i$ are given by Levi-Civita [1]

$$\gamma_{pqr} = \{{}^{p}h_l\}_m\, {}_{q}h^l\, {}_{r}h^m,$$

and satisfy the equations

$$\gamma_{pqr} + \gamma_{qpr} = 0.$$

The coefficient γ_{pqr} denotes geometrically the rate of change of the scalar product of two members ${}^{p}h_i$ and ${}^{q}h_i$ of the ennuple at a point when the first is moved by local and the second by Levi-Civita displacement along a third member ${}^{r}h_i$. These coefficients may be regarded as an ennuplet tensor and one may derive from them a covariant tensor of rank three by composition with the h's as

$$\gamma_{ijk} = \gamma_{pqr}\, {}^{p}h_i\, {}^{q}h_j\, {}^{r}h_k. \tag{3.1}$$

Now in view of the identity

$$\sum_p \left({}^{p}h_i \frac{\partial {}^{p}h_j}{\partial x^k} - {}^{p}h_i \{{}^{p}h_j\}_k \right) = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right),$$

the conditions (2.5) and (2.6) may be written as

$$\sum_p ({}^{p}h_{1|i}\{{}^{p}h_{1|j}\}_k - {}^{p}h_{2|j}\{{}^{p}h_{2|i}\}_k) = 0,$$

$$\sum_p ({}^{p}h_{1|i}\{{}^{p}h_{1|j}\}_k - {}^{p}h_{3|i}\{{}^{p}h_{3|k}\}_j) = 0. \tag{3.2}$$

If then $\gamma_{\lambda|pqr}$ denote the coefficients of rotation with respect to the ennuples ${}^{p}h_{\lambda|i}$, $\lambda = 1\ 2, 3$, the conditions (3.2) may, in consequence of (3.1) be written as

$$\gamma_{1|ijk} + \gamma_{2|ijk} = 0, \quad \gamma_{1|ijk} + \gamma_{3|kij} = 0. \tag{3.3}$$

4. As stated at the beginning of the paper, we now propose to construct types of three-dimensional spaces which are compatible with pairs of teleparallelisms satisfying the conditions (2.5) and (2.6). We shall also take into consideration the case when the parallelisms are not distinct but satisfy these conditions. For this purpose, let us consider a space of the following kind. It is a generalisation of a three-dimensional space of constant curvature arising from a generalisation of Clifford parallelism [3]. It will be convenient in the rest of the paper to denote the space coordinates by x_1, x_2, x_3

instead of by x^1, x^2, x^3 and to regard two coordinates as distinct when their indices are denoted by distinct letters. Let

$$\chi = \chi(x_1, x_2, x_3), \ \phi = \phi(x_1, x_2, x_3), \ \psi = \psi(x_1, x_2, x_3),$$

be three real analytic functions of the variables to be determined suitably. Using the rotations

$$\theta = x_1^2 + x_2^2 + x_3^2, \ E = \chi^2 + \tfrac{1}{4}\,\theta\,\phi^2\,\psi, \ F = \chi^2\,\phi^2(1 - \psi),$$

let the metric tensor of the space be defined as follows :

$$\left.\begin{aligned} &\frac{1}{\sqrt{g}} = \frac{1}{|g|^{1/2}} = E(E^2 + \theta\,F), g\,E \neq 0, \\ &g_{ii} = \frac{E^2 + x_i^2\,F}{E}\,\sqrt{g},\ g_{ij} = \frac{x_i\,x_j\,F}{E}\,\sqrt{g}. \\ \text{Therefore}\quad &g^{ii} = E^2 + (\theta - x_i^2)\,F,\ g^{ij} = -\,x_i\,x_j\,F. \end{aligned}\right\} \qquad (4.1)$$

In this space let us take the following pair of orthogonal ennuples (*)

$$\left.\begin{aligned} &{}_1h^1 = \chi^2 + \frac{x_1^2 - x_2^2 - x_3^2}{4}\,\phi^2\psi, \quad {}_1h^2 = \frac{x_1x_2}{2}\,\phi^2\psi \pm x_3\chi\phi, \quad {}_1h^3 = \frac{x_1x_3}{2}\,\phi^2\psi \mp x_2\chi\phi, \\ &{}_2h^1 = \frac{x_1x_2}{2}\,\phi^2\psi \mp x_3\chi\phi, \quad {}_2h^2 = \chi^2 + \frac{-x_1^2 + x_2^2 - x_3^2}{4}\,\phi^2\psi, \quad {}_2h^3 = \frac{x_2x_3}{2}\,\phi^2\psi \pm x_1\chi\phi, \\ &{}_3h^1 = \frac{x_1x_3}{2}\,\phi^2\psi \pm x_2\chi\phi, \quad {}_3h^2 = \frac{x_2x_3}{2}\,\phi^2\psi \mp x_1\chi\phi, \quad {}_3h^3 = \chi^2 + \frac{-x_1^2 - x_2^2 + x_3^2}{4}\,\phi^2\psi. \\ &\text{Therefore} \qquad {}^ih_i = ({}_ih^iE + x_i^2F)\sqrt{g}, \quad {}^ih_j = ({}_ih^jE + x_ix_jF)\sqrt{g}. \end{aligned}\right\} \qquad (4.2)$$

The two ennuples of the pair correspond to the upper and the lower sign in (4.2)

It is obvious that if in any space admitting an orthogonal ennuple ${}^ph_i$, e.g. in the space (4.1), we write, $a = {}_ph^t\,\dfrac{\partial\,{}^ph_s}{\partial x_j}$ then we must have

$$\left\{ {t \atop ij} \right\} = \tfrac{1}{2}\,(a + a' + \beta + \beta_e).$$

But it does not follow from this equation, regarding a as unknown, that the space admits the orthogonal ennuple.

From (4.2) we derive the following formulae which we need. The calculation is heavy but straightforward.

(*) This set of values of the ${}^ph_i$'s is a generalisation of the set given by Whittaker having the indispensable property that the $\pm$ sign must cancel out when the g_{ij}'s given by (4.1) are obtained from (4.2). Cf. Whittaker, *loc. cit.* equations (5), (10).

From (4.1) we have $\dfrac{1}{E\sqrt{g}} = E^3 + \theta F$. Write for the moment

$$G = F\left(E + \frac{1}{E^2\sqrt{g}}\right),$$

$$H_m = E\left\{E\left(\phi\frac{\partial \chi}{\partial x_m} - \chi\frac{\partial \phi}{\partial x_m}\right) - \tfrac{1}{4}\,\theta\chi\phi^3\frac{\partial \psi}{\partial x_m}\right\}.$$

Then

$$\left.\begin{aligned}
\sum_p {}^p h_i \frac{\partial^p h_i}{\partial x_i} &= -g\left[(E^3 + x_i^2 G)\frac{\partial E}{\partial x_i} + \right.\\
&\qquad \left. + (\theta - x_i^2)\left(\frac{E^2}{2}\frac{\partial F}{\partial x_i} - x_i F^2\right)\right],\\
\sum_p {}^p h_i \frac{\partial^p h_i}{\partial x_j} &= -g\left[(E^3 + x_i^2 G)\frac{\partial E}{\partial x_j} + \right.\\
&\qquad \left. + (\theta - x_i^2)\frac{E^2}{2}\frac{\partial F}{\partial x_j} + x_j F\,(E^2 + x_i^2 F)\right],\\
\sum_p {}^p h_j \frac{\partial^p h_i}{\partial x_i} &= -x_j g\left[x_i\left(G\frac{\partial E}{\partial x_i} - \frac{E^2}{2}\frac{\partial F}{\partial x_i} + x_i F^2\right) - \right.\\
&\qquad \left. - \frac{1}{E\sqrt{g}}\left(\frac{E}{2}\phi^2\psi + F\right)\right] \pm x_k g E\,[H_i + x_i F\chi\phi],\\
\sum_p {}^p h_i \frac{\partial^p h_j}{\partial x_i} &= -x_j g\left[x_i\left(G\frac{\partial E}{\partial x_i} - \frac{E^2}{2}\frac{\partial F}{\partial x_i} + x_i F^2\right) + \right.\\
&\qquad \left. + \frac{\phi^2\psi}{2\sqrt{g}}\right] \mp x_k g E\,[H_i + x_i F\chi\phi],\\
\sum_p {}^p h_i \frac{\partial^p h_j}{\partial x_k} &= -x_i x_j g\left[G\frac{\partial E}{\partial x_k} - \frac{E^2}{2}\frac{\partial F}{\partial x_k} + x_k F^2\right]\\
&\qquad \mp gE\left[x_k H_k - \left(\frac{1}{E\sqrt{g}} - x_k^2 F\right)\chi\phi\right].
\end{aligned}\right\} \quad (4.3)$$

In the above formulae (4.3) the upper and the lower sign refer to the two ennuples corresponding to the upper and the lower sign in (4.2) if ijk is an even permutation of 123. If ijk is an odd permutation of 123, these signs will have to be reversed.

Now each of the conditions (2.5) and (2.6) is a relation between two orthogonal ennuples in which the indices i, j, k may or may not be all distinct. Apply the formulae (4.3) in these conditions and make use of the obvious fact that $A \pm B = 0$ implies $A = B = 0$,

whatever quantities A and B may be. Then, after some simplifications, we obtain the following results :

The condition (2.5) is expressed by the equations

$$F(2E^2+\theta F)\left(x_i\frac{\partial E}{\partial x_j}-x_j\frac{\partial E}{\partial x_i}\right)-\frac{E^3}{2}\left(x_i\frac{\partial F}{\partial x_j}-x_j\frac{\partial F}{\partial x_i}\right)=0, \quad (4.4)$$

$$2E^2\frac{\partial E}{\partial x_j}+\theta E\frac{\partial F}{\partial x_j}+x_j\{2EF-\phi^2\psi(E^2+\theta F)\}=0. \quad (4.5)$$

The condition (2.6) is expressed by (4.4), (4.5) and

$$E\left\{E\left(\phi\frac{\partial\chi}{\partial x_j}-\chi\frac{\partial\phi}{\partial x_j}\right)-\frac{1}{4}\theta\chi\phi^3\frac{\partial\psi}{\partial x_j}\right\}+x_jF\chi\phi=0. \quad (4.6)$$

Each of these three equations represents a set of three such equations obtained by corresponding arrangements of x_i, x_j, x_k.

Along with them we introduce the equation

$$\chi\phi=0, \quad (4.7)$$

for the purpose of consideration of the case when the pair of ennuples are not distinct.

5. The above equations are partial differential equations in the three unknown functions χ, ϕ and ψ. Suitable solutions of them, if they exist, will reduce the space (4.1) to types of three-dimensional Riemannian spaces satisfying the conditions.

SPACES OF TYPE I. These spaces are given by those solutions of (4.4) and (4.5) which are not solutions of (4.6). It would follow that if corresponding to a desired solution the pair of ennuples (4.2) are denoted by ${}^{\nu}h_{1|i}$ and ${}^{\nu}h_{2|i}$, they would satisfy the condition (2.5) in the corresponding space of this type to which (4.1) reduces.

One such solution is given here. The equation (4.5) may be written in the extended form as

$$E\Big[2\chi\{2E+\theta\phi^2(1-\psi)\}\frac{\partial\chi}{\partial x_j}+\theta\phi\{E\psi+2\chi^2(1-\psi)\}\frac{\partial\phi}{\partial x_j}+$$
$$+\theta\phi^2\left(\frac{E}{2}-\chi^2\right)\frac{\partial\psi}{\partial x_j}\Big]+x_jF(2E-\theta\phi^2\psi)=0.$$

As a step towards a possible solution, we take $E = 2\chi^2$. The equation then reduces to

$$\frac{\partial}{\partial x_j}(\chi\phi) = 0, \quad \text{or} \quad \chi\phi = \text{a constant, say } l.$$

It follows that

$$F = l^2 - \frac{4\chi^4}{\theta}, \quad 2E^2 + \theta F = 4\chi^4 + l^2\theta.$$

Substituting these values in (4.4), we obtain the following solution :

$$\left.\begin{aligned} &\chi = \text{a suitable solution of the three equations} \\ &\qquad x_i \frac{\partial \chi}{\partial x_j} - x_j \frac{\partial \chi}{\partial x_i} = 0 \\ &\text{(e.g. } \chi \text{ is an algebraic function of } \theta), \\ &\qquad \chi\phi = l, \quad \psi = \frac{4\chi^4}{l^2\theta}. \end{aligned}\right\} \tag{5.1}$$

In order to obtain the equations of the pair of teleparallelisms in one of these spaces, let us take (perhaps) the simplest case when $\chi = 1$ in (5.1). It may be verified that this case does not give a solution of (4.6). Put always

$$\Gamma^i_{jk} = {}_p h^i \frac{\partial\, {}^p h_j}{\partial x_k}.$$

Then with reference to (4.1) and (4.2), calculation gives the following values :

$$\left\{ {i \atop ii} \right\} = \frac{x_i}{\theta}\left(\frac{x_i^2}{\theta} - 1\right), \quad \left\{ {i \atop jj} \right\} = \frac{x_i}{\theta}\left(\frac{x_j^2}{\theta} + 1\right),$$

$$\left\{ {i \atop ij} \right\} = \frac{x_j}{\theta}\left(\frac{x_i^2}{\theta} - 1\right), \quad \left\{ {i \atop jk} \right\} = \frac{x_i x_j x_k}{\theta^2},$$

$$\Gamma^i_{ii} = \left\{ {i \atop ii} \right\}, \quad \Gamma^i_{jj} = \left\{ {i \atop jj} \right\} \mp \frac{l x_j x_k}{2\theta},$$

$$\Gamma^i_{ij} = \left\{ {i \atop ij} \right\} \pm \frac{x_i x_k}{2l\theta}\left(\frac{4}{\theta} - l^2\right), \quad \Gamma^i_{ji} = \left\{ {i \atop ji} \right\} \mp \frac{2x_i x_k}{l\theta^2},$$

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} \pm \frac{1}{2l\theta}\left(\frac{4x_i^2}{\theta} + l^2 x_j^2\right).$$

Therefore the equations of the pair of teleparallelisms in this space are given by

$$dV^i + \sum_{jk} \left\{ {i \atop jk} \right\} V^j dx_k \pm \frac{l}{2\theta} (\Sigma V^t dx_t)(x_j dx_k - x_k dx_j) \mp$$

$$\mp \frac{2x_i}{l\theta^2} [V^i(x_j dx_k - x_k dx_j) + V^j(x_k dx_i - x_i dx_k) + V^k(x_i dx_j - x_j dx_i)] = 0, \tag{5.2}$$

where the upper and the lower sign correspond to the upper and the lower sign in (4.2) if ijk is an even permutation of 123. If ijk is an odd permutation of 123, the signs have to be reversed.

Spaces of type II. These spaces are given by those solutions of (4.4), (4.5) and (4.6) which are not solutions of (4.7). It would follow that if corresponding to a desired solution the pair of ennuples (4.2) are denoted by ${}^p h_{1|i}$ and ${}^p h_{3|i}$, they would satisfy the condition (2.6), and therefore also (2.5) when ${}^p h_{2|i} = {}^p h_{3|i}$, in the corresponding space of this type to which (4.1) reduces.

Now equation (4.6) can be written as

$$E^2 \frac{\partial}{\partial x_j}\left(\frac{\chi}{\phi}\right) - \chi\phi \left\{ \frac{1}{4}\theta E \frac{\partial \psi}{\partial x_j} - x_j \chi^2 (1 - \psi) \right\} = 0.$$

This equation is satisfied by

$$\psi = 1, \ \chi = l\phi, \text{ where } l \text{ is a constant.}$$

It follows that

$$E = \chi^2 \left(1 + \frac{\theta}{4l^2} \right), \ F = 0.$$

Substituting these values, it is seen that (4.4) is identically satisfied and (4.5) reduces to

$$\frac{\partial \chi}{\partial x_j} = 0, \quad \text{or} \quad \chi = \text{a constant, say } 1.$$

A solution in this case is therefore given by

$$\chi = \psi = 1, \ \phi = \frac{1}{l} \ . \tag{5.3}$$

Hence the corresponding space is given by

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{\{1 + (x_1^3 + x_2^2 + x_3^2)/4l^2\}^2}, \text{ a space of constant curvature } \frac{1}{l^2}.$$

In order to obtain the equations of the pair of teleparallelisms in this space, we get the following values by calculation with reference to (4.1) and (4.2) :

$$\frac{x_i}{2l^2E} = -\left\{\begin{matrix} i \\ ii \end{matrix}\right\} = \left\{\begin{matrix} i \\ jj \end{matrix}\right\} = -\Gamma^i_{ii} = \Gamma^i_{jj}$$

$$\frac{x_j}{2l^2E} = -\left\{\begin{matrix} i \\ ij \end{matrix}\right\} = -\Gamma^i_{ij} = -\Gamma^i_{ji},$$

$$0 = \left\{\begin{matrix} i \\ jk \end{matrix}\right\}, \ \pm\frac{1}{lE} = \Gamma^i_{jk}.$$

Therefore the equations of the pair of teleparallelisms are given by

$$dV^i + \sum_{jk}\left[\left\{\begin{matrix} i \\ jk \end{matrix}\right\} \pm \frac{R}{lE}\right] V^j dx_k = 0, \tag{5.4}$$

where $R = 1$ or 0 according as i, j, k are distinct or not, and the upper and the lower sign have the same significance as always stated before. The equations (5.4) are exactly the same as those given by Whittaker [3]. Hence the pair of teleparallelisms are *Clifford's right and left parallelisms*. It is interesting to notice that Clifford's parallels satisfy (2.6).

From (5.2) and (5.4) it may easily be verified that, with reference to (2.4), the property (2.3) is satisfied for these spaces of types I and II, although it does not generally hold for the space (4.1) with reference to (4.2).

Spaces of type III. These spaces are given by the solutions of (4.5) and (4.7). In view of (4.7), the equations (4.4) and (4.6) are identically satisfied. Equation (4.7) gives either $\chi = 0$ or

$\phi = 0$; both zero is ruled out. In either case, it follows from (4.2) that there is one distinct ennuple. It would follow that if corresponding to a desired solution the single ennuple (4.2) is denoted by ${}^{\nu}h_{1|i}$, it would satisfy both the conditions (2.5) and (2.6), when, ${}^{\nu}h_{3|i} = {}^{\nu}h_{2|i} = {}^{\nu}h_{1|i}$, in the corresponding space of this type to which (4.1) reduces.

(1) Taking $\chi = 0$, $\phi \neq 0$, we get $E = \frac{1}{4}\,\theta\phi^2\psi$, $F = 0$.

Substituting these values in (4.5), we obtain

$$\frac{\partial}{\partial x_j}(\phi^2\psi) = 0, \text{ or } \phi^2\psi = \text{a constant, say } 4.$$

Hence the space is given by

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{(x_1^2 + x_2^2 + x_3^2)^2}.$$

(2) Taking $\phi = 0$, $\chi \neq 0$, we obtain

$$\frac{\partial \chi}{\partial x_j} = 0, \text{ or } \chi = \text{a constant, say } 1.$$

Hence the space is given by

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

Thus in both cases we get the space of zero curvature, and the only teleparallelism in this space is the *Euclidean parallelism*.

In conclusion it may be mentioned that the types of spaces obtained in this paper have been derived, one after another, from the space (4 1) by the imposition of additional conditions and, in this sense, any one type is a generalisation of the other which succeeds it. Spaces of type I which admit a pair of teleparallelisms satisfying (2.5) seem worth investigation, the other two types being known.

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Department of Pure Mathematics
Calcutta University

ON A ONE-SIDED TAUBERIAN THEOREM; A FURTHER NOTE

By C. T. RAJAGOPAL

[Received December 8, 1952]

1. Introduction. My earlier note with the same title [3] was mainly about an extension of a theorem of Karamata on the Laplace-Abel transform to certain regular integral transforms Φ of a real function $a(u)$ satisfying the classical one-sided condition,

$$a(u)\, u \geqslant -W,\ W > 0, \text{ for } u \geqslant 1, \text{ and } a(u) = 0 \text{ for } u < 1. \quad (1)$$

In the light of a recent paper of W. B. Pennington [2], it may be of some interest to know how my extension [3, Theorem A] is affected when (1) is replaced by the condition,

$$a(u)\, u \log\log u \geqslant -W,\ W > 0, \text{ for } u \geqslant 3, \text{ and } a(u) = 0 \text{ for } u < 3, \quad (2)$$

which was introduced by Vijayaraghavan [6] as being more appropriate to certain cases of "infinite summability" of series by Abel's and $(C, 1)$ processes. Some theorems already proved by me [3, 4] and their extensions or modifications presented here (Theorems A′, B′, C′, D′) show that most of the results on infinite summability proved by Vijayaraghavan are implicit in inequalities connecting the partial sums of a series with a class of function-transforms of the series, under a Tauberian condition on the terms of the series, whether of type (1) or of type (2). Since Tauber's original theorem is itself implicit in one of these inequalities (Theorem B′), it may be presumed that the standard results on infinite summability are mostly of the same depth as Tauber's theorem.

2. Another one-sided theorem. The modification of my Theorem A of [3], due to the replacement of (1) by (2), runs as follows.

THEOREM C. *Suppose that $a(u)$ is a real function, integrable over each finite interval $0 \leqslant u \leqslant u_0$, and satisfying* (2). *Suppose that $\phi(u)$ satisfies the conditions* (C) *of my previous paper* [3, § 1, p. 47], *i.e. for $u \geqslant 0$, $\phi(u)$ is positive, twice differentiable,*

$$\phi'(u) < 0, \phi(0) = 1, \int_\epsilon^\infty \frac{\phi(u)}{u}\, du < \infty \textit{ for every } \epsilon > 0.$$

Suppose finally that

$$\Phi(t) = \int_0^\infty \phi(ut)\, a(u)\, du \quad \textit{exists for } t > 0, \tag{3}$$

$$\Phi(t) \leqslant M.$$

Then $A(h)$ defined by

$$A(h) = \int_0^h a(u)\, du \equiv \int_3^h a(u)\, du, \tag{4}$$

satisfies the inequality

$$\limsup_{h \to \infty} A(h) \leqslant M + W. \tag{5}$$

The following theorem is a deduction from the proof of the preceding and originated with Vijayaraghavan in the case $\phi(u) = e^{-u}$ [6, Theorem 3].

THEOREM c. *If, in Theorem* C, *the hypothesis on $\Phi(t)$ in* (3) *is replaced by*

$$\Phi(t) \to -\infty \textit{ as } t \to +0,$$

then the conclusion (5) *will be changed to*

$$A(h) \to -\infty \textit{ as } h \to \infty.$$

Pennington has in effect proved the case $\phi(u) = e^{-u}$ of Theorem C [2, p. 563], by an appeal to another similar theorem of his [2, Theorem 5], and shown that (5) is "the most precise inequality of this form that we can obtain" under the Tauberian condition (2). I offer a direct proof of Theorem C, slightly modifying the arguments in the case in which the Tauberian condition is (1) instead of (2) [3, proof of Theorem A], and stressing once again the motivation presented in those arguments.

Proof of Theorem C. We first obtain an inequality for $A(h)$. $h > 3$, defined by (4), using (3) and (2).

$$M > \int_0^\infty \phi(ut)\, a(u)\, du$$

$$\geqslant \int_3^h \phi(ut) \left[a(u) + \frac{W}{u \log\log u} \right] du - \int_3^h \frac{W}{u \log\log u}\, du +$$

$$+ \int_h^\infty \phi(ut)\, a(u)\, du$$

$$> \phi(ht) \int_3^h \left[a(u) + \frac{W}{u \log\log u} \right] du - \int_3^h \frac{W}{u \log\log u}\, du -$$

$$- \int_h^\infty \phi(ut) \frac{W}{u \log\log u}\, du$$

$$> \phi(ht)\, A(h) + \Big[\phi(ht) - 1 \Big] W \int_3^h \frac{du}{u \log\log u} -$$

$$- \frac{W}{\log\log h} \int_h^\infty \frac{\phi(ut)}{u}\, du. \qquad (6)$$

Now

$$\int_3^h \frac{du}{u \log\log u} = \Big[1 + o(1) \Big] \frac{\log h}{\log\log h}, \quad h \to \infty, \qquad (7)$$

$$\int_h^\infty \frac{\phi(ut)}{u}\, du = \left[\log ht + \int_{ht}^\infty \frac{\phi(u)}{u}\, du \right] - \log ht$$

$$= O(1) - \log ht, \quad ht \to 0. \qquad (8)$$

Corresponding to every h, sufficiently large, we can choose $t > 0$ so that ht is arbitrarily small, with the result that (7) and (8) hold simultaneously. Under these conditions, we can use (7) and (8) in (6), obtaining

$$M > \phi(ht)\, A(h) + \Big[\phi(ht) - 1 \Big] \Big[W + o(1) \Big] \frac{\log h}{\log\log h} +$$

$$+ \frac{O(1) + W \log ht}{\log\log h},$$

or

$$A(h) < \frac{M - (W \log t/\log \log h)}{\phi(ht)} - \frac{W \log h}{\log \log h} +$$

$$+ \frac{1 - \phi(ht)}{\phi(ht)} o \left(\frac{\log h}{\log \log h}\right) + \frac{O(1)}{\phi(ht) \log \log h}$$

$$(h \to \infty, ht \to 0),$$

i.e.

$$A(h) < g(t) + O(ht) . o \left(\frac{\log h}{\log \log h}\right) + \frac{O(1)}{\phi(ht) \log \log h}, \tag{9}$$

where

$$g(t) = \frac{M - (W \log t/\log \log h)}{\phi(ht)} - \frac{W \log h}{\log \log h}. \tag{10}$$

If it is possible to choose t so that $g(t)$ defined by (10) has the least value in terms of any sufficiently large h and simultaneously the other terms on the right side of (9) are negligible, then that value of $g(t)$ substituted in (9) will obviously give the best form of (9). From certain considerations for locating the minimum, under the conditions stated, of a function $f(t)$ related to $g(t)$ [3, pp. 51-52], it follows that the least value of $g(t)$ under the same conditions corresponds to

$$h \sim \frac{1}{\phi'(0) \, t \log t}, \quad h \to \infty, \tag{11}$$

and (as can be easily verified) is such that

$$\left.\begin{aligned} & ht \frac{\log h}{\log \log h} = o(1), \\ g(t) & = \frac{M - (W \log ht/\log \log h)}{\phi(ht)} + \frac{1 - \phi(ht)}{\phi(ht)} \frac{W \log h}{\log \log h} \\ & = \frac{M - W[-1 + o(1)]}{1 + o(1)} + o(1). \end{aligned}\right\} \tag{12}$$

The use of (12) in (9) establishes the conclusion (5) of Theorem C.

3. Analogous theorems for series and their (Φ, λ)-transforms. Let $\sum_{n=1}^{\infty} a_n$ a be real series. Then we define the (Φ, λ)-transform of the series by

$$\Phi_\lambda(t) = \sum_1^\infty a_n \phi(\lambda_n t), t > 0,\ 0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots, \lambda_n \to \infty \quad (13)$$

where

$$\phi(u) \equiv \int_u^\infty \psi(x)\, dx,$$

$\psi(u)$ is bounded in $(0, \delta)$ for some $\delta > 0$, and satisfies the set of standard conditions :

$$\psi(u) \geqslant 0, \int_0^\infty \psi(u)\, du = 1, \int_0^\infty \psi(u) |\log u| du < \infty. \quad (C^*)$$

It is clear that the (Φ, λ)-transform of Σa_n is regular and that $\phi(u)$ can be not only one of the kernels of Laplace-Abel, Stieltjes and Lambert [3, p. 49] but also the kernel of Riesz :

$$\phi(u) = (1-u)^k, k > 0, \text{ for } u \leqslant 1, \text{ and } \phi(u) = 0 \text{ for } u > 1. \quad (14)$$

All the theorems of this note and its predecessor [3] can be restated for the (Φ, λ)-transform of a series. The restatements of the major Theorems A, B, C are contained in the Theorems A′, B′, C′ which follow. Theorem B′ is known in more general forms [e.g. 5 and references therein]; it is stated here for comparison with Theorems A′, C′.

Theorem B′. *In* (13), *let*

$$|a_n| \leqslant W \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \textit{ for } n \geqslant 1, \frac{\lambda_{n-1}}{\lambda_n} \sim 1 \textit{ as } n \to \infty.$$

Then $A_n \equiv a_1 + a_2 + \dots + a_n$ *satisfies the inequality*

$$\limsup_{n\to\infty} |A_n - \Phi_\lambda(t)| \leqslant W \left[\int_0^q \frac{1-\phi(u)}{u} du + \int_q^\infty \frac{\phi(u)}{u} du \right],$$

where $t = \dfrac{q}{\lambda_n}$, $q > 0$.

Theorem A′. *In* (13), *let*

$$a_n \geqslant -W \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}, \ W > 0, \textit{ for } n \geqslant 1. \quad (1')$$

Then $A_n \equiv a_1 + a_2 + \ldots + a_n$ *satisfies the inequality*

$$\limsup_{n\to\infty} [\phi(\lambda_n t) A_n - W \log\log \lambda_n - \Phi_\lambda(t)] \leqslant W(q\bar{\psi}_0 - \log q - \gamma), \quad (4')$$

where

$$t = \frac{q}{\lambda_n \log \lambda_n}, \quad q > 0, \tag{15}$$

$$\bar{\psi}_0 = \limsup_{u\to+0} \psi(u),$$

$$\gamma = -\lim_{t\to+0} \left[\int_t^\infty \frac{\phi(u)}{u}\, du + \log t \right].$$

When $\bar{\psi}_0 \neq 0$, *the best form of* (4′) *is that with the least value of the right-hand member of* (4′) *corresponding to* $q = 1/\bar{\psi}_0$. *If we take this least value and further assume that*

$$\limsup_{t\to+0} \Phi_\lambda(t) < \infty, \tag{3'}$$

then (4′) *can be written as*

$$\limsup_{n\to\infty} [A_n - W \log\log \lambda_n - \Phi_\lambda(t)] \leqslant W\,(1 + \log \bar{\psi}_0 - \gamma).$$

Theorem A′ is proved elsewhere [4] in a slightly less general form. The passage from the proof of Theorem A in [3] to the proof of Theorem A′ requires merely the use of the following lemma besides some obvious changes one of which is the replacement of (11) in the proof of Theorem A by (15), suggested in fact by (11).

LEMMA. (i) *The existence of* γ *in Theorem* A′ *follows from that of the integrals*

$$\int_0^1 \frac{1 - \phi(u)}{u}\, du = \int_0^1 \psi(u) \log \frac{1}{u}\, du,$$

$$\int_1^\infty \frac{\phi(u)}{u}\, du = \int_1^\infty \psi(u) \log u\, du.$$

(ii) *If* (15) *holds, then*

$$\sum_{\nu=1}^{n} [\phi(\lambda_\nu t) - \phi(\lambda_n t)] \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu} < q\bar{\psi}_0 + o(1), \quad t \to +0.$$

PROOF. The proof of (i) is obvious; that of (ii) is as follows. Supposing (as we may) that $\lambda_n t < \delta$, we have

$$\sum_{\nu=1}^{n} [\phi(\lambda_\nu t) - \phi(\lambda_n t)] \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu} = \sum_{\nu=1}^{n} \left[\int_{\lambda_\nu t}^{\lambda_n t} \psi(u)\, du \right] \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}$$

$$\leqslant \sum_{\nu=1}^{n} (\lambda_n t - \lambda_\nu t) \overline{\underset{\lambda_\nu t \leqslant u \leqslant \lambda_n t}{\text{bound}}}\, \psi(u) . \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}$$

$$< \lambda_n t \overline{\underset{0 \leqslant u \leqslant \lambda_n t}{\text{bound}}}\, \psi(u) \sum_{\nu=1}^{n} \left[1 + \int_{\lambda_1}^{\lambda_n} \frac{du}{u} \right].$$

The result (ii) of our lemma follows when we let $t \to +0$.

THEOREM C′. *In the general case of Theorem* A′, *let* (1′) *be replaced by*

$$a_n \geqslant - W \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \log \log \lambda_n}, \; W > 0, \textit{for } \lambda_n \geqslant 3. \tag{2'}$$

Then (4′) *will be altered to*

$$\limsup_{n \to \infty} [\phi(\lambda_n t) A_n - \Phi_\lambda(t)] \leqslant W(q \bar{\psi}_0 + 1), \tag{5'}$$

where

$$t = \frac{q \log \log \lambda_n}{\lambda_n \log \lambda_n}, \quad q > 0.$$

If we take q arbitrarily small and further assume that (3′) *holds, then* (5′) *assumes the simpler form*

$$\limsup_{n \to \infty} [A_n - \Phi_\lambda(t)] \leqslant W.$$

The proof of Theorem C′ may be obtained from that of Theorem C, more or less as the proof of Theorem A′ from that of Theorem A. Theorem C′ is a generalization of one of Pennington for the case $\phi(u) = e^{-u}$, $\lambda_n = n$ [2, Theorem 5]. Pennington's other theorems [2, Theorems 3, 4] admit of similar generalizations.

4. A special theorem for the Riesz transform of a series. The next theorem is in a direction opposite to Theorem A′,

but it treats only the case in which the (Φ, λ)-transform is the (R, λ, k)-transform.

THEOREM D′. *In* (13), *let* $\Phi(u)$ *be the Riesz kernel of* (14), *and let* (1′) *hold. Then* $A_n = a_1 + a_2 + \dots + a_n$ *satisfies the inequality*

$$\liminf_{n\to\infty} [A_n - \Phi_\lambda(\lambda_n^{-1})] \geqslant -W(q\bar{\psi}_q - \log q - 1),$$

where

$$0 < q < 1, \qquad \bar{\psi}_q = \overline{\text{bound}}_{0\leqslant u\leqslant q}\, \psi(u) = \begin{cases} \psi(0) \text{ when } k \geqslant 1, \\ \psi(q) \text{ when } 0 < k < 1. \end{cases}$$

Included in Theorem D′ is the following which generalizes a result of Vijayaraghavan for $\lambda_n = n$, $k = 1$ [6, Theorem 2].

THEOREM d′. *In the special case*

$$\lim_{t\to+0} \Phi_\lambda(t) = \infty,$$

of Theorem D′, *the conclusion will assume the more precise form*

$$\lim_{n\to\infty} A_n = \infty.$$

PROOF OF THEOREM D′. With $t = \lambda_n^{-1}$ we find, from the definition of $\Phi_\lambda(t)$ and (1′),

$$\begin{aligned} A_n - \Phi_\lambda(t) &= \sum_{\nu=1}^{n} [1 - \phi(\lambda_\nu t)]\, a_\nu \\ &\geqslant -\sum_{\nu=1}^{n} [1 - \phi(\lambda_\nu t)]\, W\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu} \\ &\geqslant -\Big\{ \sum_{\lambda_\nu t\leqslant q} + \sum_{q<\lambda_\nu t\leqslant \lambda_n t} \Big\} [1 - \phi(\lambda_\nu t)]\, W\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}. \qquad (16) \end{aligned}$$

Now

$$\begin{aligned} \sum_{\lambda_\nu t\leqslant q} [1 - \phi(\lambda_\nu t)]\, W\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu} &= W \sum_{\lambda_\nu t\leqslant q} \Big[\int_0^{\lambda_\nu t} \psi(u)\, du \Big] \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu} \\ &\leqslant W \sum_{\lambda_\nu t\leqslant q} \lambda_\nu t\, \overline{\text{bound}}_{0\leqslant u\leqslant \lambda_\nu t}\, \psi(u)\cdot \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu} \\ &\leqslant W\bar{\psi}_q q; \qquad (17) \end{aligned}$$

$$\sum_{q<\lambda_\nu t \leqslant \lambda_n t} [1-\phi(\lambda_\nu t)]\, W \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu} < \sum_{q<\lambda_\nu t \leqslant \lambda_n t} W \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}$$

$$< W\left[\frac{\lambda_m - \lambda_{m-1}}{\lambda_m} + \sum_{\nu=m+1}^{n} \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}\right],$$

where m is such that $\lambda_{m-1} t \leqslant q < \lambda_m t$. Hence

$$\sum_{q<\lambda_\nu t \leqslant \lambda_n t} [1-\phi(\lambda_\nu t)]\, W \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu} < W\left[\frac{\lambda_m - \lambda_{m-1}}{\lambda_m} + \int_{\lambda_m}^{\lambda_n} \frac{dx}{x}\right]$$

$$< W + W \log \frac{\lambda_n}{\lambda_m}$$

$$< W + W \log \frac{1}{q}. \tag{18}$$

By using (17) and (18) in (16), and letting $\lambda_n = t^{-1} \to \infty$, we reach the conclusion of Theorem D′.

Remarks on Theorems D′, d′. (i) *Theorem* d′ *is significant in that* ∞ *in its enunciation cannot be replaced by a finite* s, as shown by a well-known example of Ananda-Rau [1].

(ii) The proofs of Theorems D′, d′ depend on the fact that, in the (R, λ, k)-transform, $\phi(u) = 0$ for $u > 1$. Hence they are inapplicable to the cases in which $\phi(u)$ is one of the kernels of Laplace-Abel, Stieltjes and Lambert. In fact, certain results of Vijayaraghavan [6, Theorems 5, 6] go to show that Theorems D′, d′ cannot be extended to cover these cases.

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Ramanujan Institute of Mathematics (Karaikudi)
Madras

A THEOREM ON CONGRUENCES

By L. CARLITZ

[Received December 25, 1952]

1. Let $f(x)$ denote a polynomial with rational integral coefficients and discriminant D. In this note we prove the theorem that if

$$f(x) \equiv 0 \pmod{p^r}, \tag{1.1}$$

is solvable for $r = \delta + 1$, where p^δ is the highest power of the prime p dividing D, then the congruence is solvable for all r. While this theorem is contained in a more general theorem of Hensel [1, p. 68], a direct proof seems of interest. Also Theorem 1 below is perhaps of interest in itself.

2. Let $D = p^\delta D_0$, where $(p, D_0) = 1$, and let the integer a satisfy the congruence

$$f(a) \equiv 0 \pmod{p^{\delta+1}}. \tag{2.1}$$

It follows that

$$f(x) \equiv (x - a)\, g(x) \pmod{p^{\delta+1}}, \tag{2.2}$$

where $g(x)$ denotes a polynomial with integral coefficients. Consequently

$$f'(a) \equiv g(a) \pmod{p^{\delta+1}}. \tag{2.3}$$

Now let p^τ be the highest power of p dividing $f'(a)$. Assume $2\tau > \delta$, so that

$$f'^2(a) \equiv 0 \pmod{p^{\delta+1}}. \tag{2.4}$$

We shall show that (2.4) leads to a contradiction.

Let $R(h(x), k(x))$ denote the resultant of the polynomials $h(x)$, $k(x)$, and let $D(h(x))$ denote the discriminant of $h(x)$. Then in the first place

$$R(x - a, g(x)) = g(a). \tag{2.5}$$

Secondly, since [1, p. 60]

$$D(h(x)\, k(x)) = (-1)^{rs} D(h(x))\, D(k(x))\, R^2(h(x), k(x)),$$

where $r = \deg h(x)$, $s = \deg k(x)$, we have in particular

$$\begin{aligned} D((x-a)\,g(x)) &= (-1)^{n-1} D(x-a) D(g(x)) R^2(x-a, g(x)) \\ &= (-1)^{n-1} D(g(x)) g^2(a) \\ &\equiv (-1)^{n-1} D(g(x)) f'^2(a) \pmod{p^{\delta+1}}, \end{aligned} \tag{2.6}$$

by (2.5) and (2.3). Thus, using (2.2), (2.6) becomes

$$D(f(x)) \equiv (-1)^{n-1} D(g(x)) f'^2(a) \pmod{p^{\delta+1}}. \tag{2.7}$$

Comparing (2.7) with (2.4) we get at once $D(f) \equiv 0 \pmod{p^{\delta+1}}$, which contradicts the definition of δ. We have therefore proved

THEOREM 1. *Let a satisfy* (2.2) *and let p^τ be the highest power of p dividing $f'(a)$. Then*

$$2\tau \leqslant \delta. \tag{2.8}$$

3. We now suppose that a satisfies (1.1) for some $r \geqslant \delta + 1$, and show that a solution can be found $\pmod{p^{r+1}}$. Consider the congruence

$$f(a + tp^\sigma) \equiv 0 \pmod{p^{r+1}}. \tag{3.1}$$

If we suppose $2\sigma \geqslant r + 1$, then (3.1) reduces to

$$f(a) + tp^\sigma f'(a) \equiv 0 \pmod{p^{r+1}},$$

or what is the same thing

$$p^{-r} f(a) + p^{\sigma - r} t f'(a) \equiv 0 \pmod{p}. \tag{3.2}$$

Take $\sigma = r - \tau$. Then

$$2\sigma = 2r - 2\tau \geqslant r + \delta + 1 - 2\tau \geqslant r + 1,$$

by (2.8). But $2\sigma \geqslant r + 1$ is precisely the condition assumed in deriving (3.2). Moreover we see that the coefficient $p^{\sigma - r} f'(a)$ in (3.2) is prime to p ; consequently t is uniquely determined $\pmod{p}$ and the congruence (3.1) is satisfied.

We may state

THEOREM 2. *If the congruence*

$$f(x) \equiv 0 \pmod{p^r}, \tag{3.3}$$

is solvable for $r = \delta + 1$, then it is solvable for all r.

4. In connection with Theorem 2, it may be of interest to quote the following theorem proved by Nagell [2, p. 89]. We use the notation of the present paper.

The congruence (3.3), *where* $r \geqslant 2\delta + 1$, *has exactly the same number of solutions as the congruence*

$$f(x) \equiv 0 \pmod{p^{2\delta+1}}.$$

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Duke University
Durham, N. C., U. S. A.

SERIES-TO-SERIES QUASI-HAUSDORFF TRANSFORMATIONS

By M. S. RAMANUJAN†

[Received November 6, 1952 and, in revised form, April 8, 1953]

INTRODUCTION. Let $U = \sum_{i=0}^{\infty} u_i$ be a series, convergent or not, and let (a_{nk}) be an infinite square matrix. If for $n = 0, 1, 2, \ldots$ the series $\Sigma a_{nk} u_k$ converges to some value v_n, and if also the series Σv_n converges, then U is said to be summable by the *series-to-series transformation* $A = (a_{nk})$ to the sum Σv_n. If $\Sigma v_n = \Sigma u_n$ whenever the latter series converges, then the transformation is said to be regular and the matrix A is said to be an *a-matrix.*

We define the matrix δ by $\delta_{nk} = (-1)^k \binom{n}{k}$, i.e. by

$$\delta = \left| \begin{array}{ccccccc} 1 & 0 & 0 & . & . & . & . \\ 1 & -1 & 0 & . & . & . & . \\ 1 & -2 & 1 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{array} \right|$$

Let δ^* denote the transpose of δ.

It is easily seen that $\delta\delta = I = \delta^* \delta^*$. If μ is a diagonal matrix with diagonal elements $\mu_0, \mu_1, \ldots, \mu_n \ldots$ then the matrix $H = \delta \mu \delta$ is called a Hausdorff matrix and is denoted by (H, μ), and the matrix $\delta^* \mu \delta^*$ is called a quasi-Hausdorff matrix and is denoted by (H^*, μ). It can be verified that (H^*, μ) is the transpose of (H, μ).

In section 1 of this paper, necessary and sufficient conditions are obtained for a quasi-Hausdorff series-to-series transformation to be regular, i.e. for (H^*, μ) to be an a-matrix. In section 2

†The author, Government of India Junior Research Scholar at Annamalai University, wishes to thank Dr. V. Ganapathi Iyer for his helpful guidance and the referee for suggestions concerning the paper.

it is proved by using a theorem of Vermes [5] that every immediate continuation of an analytic function is a quasi-Hausdorff transformation, and that some of the results proved by Vermes follow from this fact.

1. We first prove a preliminary result which will be used to prove the main Theorem 2.

THEOREM 1. *A real sequence μ_n satisfies the condition*

$$\sup_n \sum_{k=n}^{\infty} \left| \binom{k}{n} \Delta^{k-n} \mu_{n+1} \right| < \infty, \tag{1.1}$$

if and only if μ_n is a moment constant††.

PROOF. Hausdorff [3] has proved that μ_n is a moment constant if and only if $\mu_n = \alpha_n - \beta_n$, where α_n and β_n are totally monotone, i.e. $\Delta^p \alpha_n \geqslant 0$, $\Delta^p \beta_n \geqslant 0$, $(p, n = 0, 1, 2, \dots)$.

Therefore it is sufficient to prove that condition (1.1) is satisfied if and only if $\mu_n = \alpha_n - \beta_n$, where α_n and β_n are totally monotone.

Suppose that condition (1.1) is satisfied. Then

$$\begin{aligned}\Delta^p \mu_n = \mu_{n,p} &= \mu_{n+1,p} + \mu_{n,p+1} \\ &= \mu_{n+1,p} + \mu_{n+1,p+1} + \mu_{n,p+2} \\ &= \mu_{n+1,p} + \mu_{n+1,p+1} + \mu_{n+1,p+2} + \mu_{n,p+3} \\ &= \dots;\end{aligned}$$

since (1.1) implies that $\mu_{n,p+r} \to 0$ as $r \to \infty$ and that the series $\sum_r \mu_{n+1,p+r}$ converges absolutely, these indentities imply that

$$\Delta^p \mu_n = \sum_{r=0}^{\infty} \mu_{n+1,p+r}. \tag{1.2}$$

From (1.2) it follows that

$$|\mu_{n,p}| \leqslant \sum_{r=0}^{\infty} |\mu_{n+1,p+r}|. \tag{1.3}$$

Now let

$$\mu_{n,p,r} = \sum_{m \geqslant r} \binom{m}{r} \mu_{n+r+1,p+m-r},$$

††For definitions of moment constants, difference symbols Δ^p etc. see Hardy [2] Chapter XI.

and $$\mu^*_{n,p,r} = \sum_{m\geqslant r} \binom{m}{r} |\mu_{n+r+1,\,p+m-r}|.$$

Then

$$\mu_{n,p,r} = \mu_{n+r+1,\,p} + (r+1)\,\mu_{n+r+1,\,p+1} + \\ + \frac{(r+2)\,(r+1)}{1.2}\,\mu_{n+r+1,\,p+2} + \cdots$$

Consider the double series

$$\mu_{n+r+2,p} + \mu_{n+r+2,\,p+1} + \mu_{n+r+2,\,p+2} + \cdots + \\ + (r+1)\,\mu_{n+r+2,\,p+1} + (r+1)\,\mu_{n+r+2,\,p+2} + \cdots + \\ + \frac{(r+2)\,(r+1)}{1.2}\,\mu_{n+r+2,\,p+2} + \cdots + \\ + \cdots$$

The formal sum of this by rows is $\mu_{n,p,r}$ by virtue of (1.2). The sum by columns is

$$\mu_{n+r+2,\,p} + (r+2)\,\mu_{n+r+2,\,p+1} + \frac{(r+3)\,(r+2)}{1.2}\,\mu_{n+r+2,\,p+2} + \cdots \\ = \mu_{n,p,r+1},$$

by definition. Replacing each term in the above double series by its modulus and summing by columns, we find that the sum of the new series is

$$\sum_{m\geqslant r+1} \binom{m}{r+1} |\mu_{n+r+2,\,p+m-r-1}|,$$

and because

$$\binom{m}{r+1} \leqslant \binom{m+p}{r+1} \leqslant \binom{m+p+n}{r+1+n},$$

this sum is not greater than

$$\sum_{m\geqslant r+1} \binom{p+n+m}{n+r+1} |\mu_{n+r+2,\,p+m-r-1}|,$$

a quantity which by hypothesis (1.1) has a bound independent of n, p and r. It follows that

the quantities $\mu^*_{n,p,r}$, form a bounded set, (1.3.1)

and

the double series considered above converges absolutely. (1.3.2)

The summation of the double series by columns is therefore justified, and therefore

$$\mu_{n,p,r} = \mu_{n,p,r+1}.$$

Therefore

$$\mu_{n,\,p,\,r} = \mu_{n,\,p,\,0} = \mu_{n,\,p}. \tag{1.4}$$

Also

$$\mu^*_{n,\,p,\,r} \geqslant |\mu_{n,\,p,\,r}| = |\mu_{n,\,p}|. \tag{1.5}$$

Now

$$\begin{aligned}\mu^*_{n,\,p,\,r} &= |\mu_{n+r+1,\,p}| + (r+1)|\mu_{n+r+1,\,p+1}| + \frac{(r+2)\,(r+1)}{1.2}\times \\ &\qquad \times|\mu_{n+r+1,\,p+2}| + \cdots \\ &\leqslant |\mu_{n+r+2,\,p}| + |\mu_{n+r+2,\,p+1}| + |\mu_{n+r+2,\,p+2}| + \cdots + \\ &\qquad + (r+1)\left\{|\mu_{n+r+2,\,p+1}| + |\mu_{n+r+2,\,p+2}| + \cdots\right\} + \\ &\qquad + \frac{(r+2)\,(r+1)}{1.2}\left\{|\mu_{n+r+2,\,p+2}| + \cdots\right\} + \\ &\qquad + \cdots\end{aligned}$$

by virtue of (1.3).

Since this double series has only positive terms it may be summed by columns. Therefore its sum is $\mu^*_{n,p,r+1}$, and it follows that

$$\mu^*_{n,\,p,\,r} \leqslant \mu^*_{n,\,p,\,r+1}.$$

This fact together with (1.3.1) gives

$$\mu^*_{n,\,p,\,r} \to \lim_{r\to\infty} \mu^*_{n,\,p,\,r} = \mu^*_{n,\,p}\,,\ \text{say}. \tag{1.6}$$

The above relation along with (1.5) gives

$$\mu_{n,\,p} \leqslant |\mu_{n,\,p}| \leqslant \mu^*_{n,\,p,\,r} \leqslant \mu^*_{n,\,p}. \tag{1.7}$$

In particular

$$|\mu_n| = |\mu_{n,0}| \leqslant \mu^*_n,\ \text{say}. \tag{1.8}$$

Making use of the fact that $\mu^*_{n,p,r}$ converges (as $r \to \infty$), and writing out the series involved, we can easily verify that

$$\mu^*_{n,\,p,\,r+1} - \mu^*_{n+1,\,p,\,r} = \mu^*_{n,\,p+1,\,r+1}.$$

Letting $r \to \infty$, we see that $\mu^*_{n,\,p} - \mu^*_{n+1,\,p} = \mu^*_{n,\,p+1}$, i.e. $\Delta\,\mu^*_{n,\,p} = \mu^*_{n,\,p+1}$. It follows that

$$\mu^*_{n,\,p} = \Delta\mu^*_{n,\,p-1} = \cdots = \Delta^p\,\mu^*_n.$$

Therefore it follows from (1.7) that

$$|\Delta^p \mu_n| = |\mu_{n,p}| \leqslant |\mu^*_{n,p}| = \Delta^p \mu^*_n,$$

and if we write

$$\alpha_n = \tfrac{1}{2} [\mu^*_n + \mu_n], \text{ and } \beta_n = \tfrac{1}{2} [\mu^*_n - \mu_n],$$

then

$$\mu_n = \alpha_n - \beta_n, \quad \Delta^p \alpha_n \geqslant 0, \quad \Delta^p \beta_n \geqslant 0.$$

This proves that the condition is necessary. To prove it is sufficient we observe that

$$\sum_{k\geqslant n} \binom{k}{n} |\Delta^{k-n} \mu_{n+1}| = \sum_{k\geqslant n} \binom{k}{n} |\Delta^{k-n} \alpha_{n+1} - \Delta^{k-n} \beta_{n+1}|$$
$$\leqslant \sum_{k\geqslant n} \binom{k}{n} |\Delta^{k-n} \alpha_{n+1}| + \sum_{k\geqslant n} \binom{k}{n} |\Delta^{k-n} \beta_{n+1}|$$
$$= \alpha_0 + \beta_0.$$

This proves the theorem.

THEOREM 2. *The quasi-Hausdorff matrix* (H^*, μ) *is an* α-*matrix if and only if*

(1) μ_n *is a moment constant, and* (2) $\mu_0 = 1$.

PROOF. It has been proved by Vermes [5] that the matrix $A \equiv (a_{nk})$ is an α-matrix if and only if the matrix $G \equiv g_{nk}$ defined by $g_{nk} = a_{0k} + a_{1k} + \dots + a_{nk}$ is a γ-matrix†, i.e. if and only if

$$(a) \ \sup_n \sum_{k=0}^{\infty} |g_{n,k} - g_{n,k+1}| < \infty,$$

and

$$(b) \ \lim_{n\to\infty} g_{nk} = 1 \text{ for all } k.$$

Taking A to be the matrix (H^*, μ), we have

$$a_{nk} = \binom{k}{n} \Delta^{k-n} \mu_n \ (k \geqslant n),$$
$$= 0, \qquad (k < n).$$

So

$$g_{nk} - g_{n,k+1} = a_{0,k} + a_{1,k} + \dots + a_{nk} -$$
$$- (a_{0,k+1} + a_{1,k+1} + \dots + a_{n,k+1})$$

†For the definition of γ-matrix, as well as β-matrix, see Cooke [1] pp. 65-68.

$$= \sum_{m=0}^{n} (a_{mk} - a_{m,k+1})$$

$$= \sum_{m=0}^{n} \left\{ \binom{k}{m} \Delta^{k-m} \mu_m - \binom{k+1}{m} \Delta^{k+1-m} \mu_m \right\}$$

$$= \sum_{m=0}^{n} \left\{ \binom{k}{m} \Delta^{k-m} \mu_m - \binom{k}{m} \Delta^{k+1-m} \mu_m - \binom{k}{m-1} \Delta^{k+1-m} \mu_m \right\}$$

$$= \sum_{m=0}^{n} \left\{ \binom{k}{m} \Delta^{k-m} \mu_m - \left[\binom{k}{m} \Delta^{k-m} \mu_m - \binom{k}{m} \Delta^{k-m} \mu_{m+1} \right] - \right.$$

$$\left. - \binom{k}{m-1} \Delta^{k-(m-1)} \mu_m \right\}$$

$$= \sum_{m=0}^{n} \left\{ \binom{k}{m} \Delta^{k-m} \mu_{m+1} - \binom{k}{m-1} \Delta^{k-(m-1)} \mu_m \right\}$$

$$= \binom{k}{n} \Delta^{k-n} \mu_{n+1},$$

and so condition (a) reduces to condition (1.1).

It follows from Theorem 1, that condition (a) is satisfied if and only if μ_n is a moment constant.

To prove condition (2) we note that $g_{n+p,n} = g_{n,n}$, for $p = 0, 1, 2, \ldots$, since $a_{nk} =$ for $n > k$. Since $g_{kk} = \sum_{m=0}^{k} a_{mk} = \mu_0$ (see Hausdorff [3] p. 79), the proof of Theorem 2 is complete.

NOTE. We observe that the condition for (H^*, μ) to be an α-matrix is slightly weaker than the condition for μ_n to be a regular moment constant. It is not required that the function $\chi(x)$ which generates the moment constants be continuous at $x = 0$, i.e. $\lim_{k\to\infty} a_{0,k} = 0$.

2. It has been proved by Vermes [5] that every immediate analytic continuation is a series-to-series transformation by the matrix $A \equiv (a_{nk})$,

$$a_{nk} = \binom{k}{n} p^n (1-p)^{k-n},$$

which can be easily seen to be the $[H^*, \mu]$ matrix with $\mu_n = p^n$, i.e. the transpose of the Toeplitz matrix representing the Euler transformation $E(p)$. This leads us on to the fact $A(p). A(q) = A(pq)$, where $A(p)$ is the above matrix. Also by virtue of Theorem 2, we can prove that $A(p)$ is an α-matrix if and only if $0 \leqslant p \leqslant 1$.

We define a matrix which transforms every convergent series into another convergent series, with sums not necessarily the same, to be an α_c-matrix. With this definition of an α_c-matrix and by virtue of Theorem 1, we can see that the matrix (H^*, μ) is an α_c-matrix, if and only if μ_n is a moment constant.

Hausdorff [3, 4] has proved that the matrix (H, μ) is a T-matrix if and only if μ_n is a regular moment constant, and it is a k-matrix if and only if μ_n is a moment constant. In the light of the above two theorems and what has been proved in this paper, we see that the necessary and sufficient condition that (H, μ) is a k-matrix is that its transpose is an α_c-matrix. Moreover the transpose (H^*, μ) of a T-matrix (H, μ) is an α-matrix, but the converse is not true unless the function $\chi(x)$ which generates the μ_n's is continuous at $x = 0$.

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Annamalai University

A TAUBERIAN THEOREM FOR THE TYPE OF AN ENTIRE FUNCTION

By T. V. LAKSHMINARASIMHAN

[Received February 9, 1953]

Theorem 1 and Corollary 1 of this note are refinements of a Tauberian theorem of R. C. Buck [2, p. 148]. From these I deduce Theorem 2 and Corollary 2 which are known to complete two results on the type of an integral function obtained by him [2, Corollaries 1, 2].

THEOREM 1. *Let $\phi(x)$ be positive, continuous, and for some $c > 0$, $\phi(x) \sim c/x$ as $x \to \infty$. Let $f(x)$ be non-decreasing, at least for all large x. Define*

$$\psi(x) = \exp \int_0^x \phi(t)\, dt, \quad g(x) = \int_0^x f(t)\, \phi(t)\, dt,$$

$$A = \limsup_{x\to\infty} \frac{f(x)}{\psi(x)}, \quad a = \liminf_{x\to\infty} \frac{f(x)}{\psi(x)}.$$

Then, if $\lambda \geqslant 1$, we have

$$\limsup_{x\to\infty} \frac{g(x)}{\psi(x)} \geqslant \frac{a}{\lambda^c} + \frac{A \log \lambda^c}{\lambda^c}, \tag{1}$$

$$\liminf_{x\to\infty} \frac{g(x)}{\psi(x)} \leqslant \frac{A}{\lambda^c} + a \log \lambda^c. \tag{2}$$

PROOF. We have, from the definition of g and ψ,

$$g(\lambda x) = \int_0^x \frac{f(t)}{\psi(t)} \psi'(t)\, dt + \int_x^{\lambda x} \frac{f(t)}{\psi(t)} \psi'(t)\, dt.$$

We can choose x_0 so that $f(x)$ is non-decreasing for $x \geqslant x_0$. With this restriction on x,

$$\frac{g(\lambda x)}{\psi(\lambda x)} \geqslant \frac{1}{\psi(\lambda x)} \int_0^x \frac{f(t)}{\psi(t)} \psi'(t)\, dt + \frac{f(x)}{\psi(\lambda x)} \int_x^{\lambda x} \frac{\psi'(t)}{\psi(t)}\, dt$$

$$= I + J, \quad \text{say},$$

where

$$I = \frac{\dfrac{\psi(x)}{\psi(\lambda x)}\left[\displaystyle\int_0^x \frac{f(t)}{\psi(t)}\psi'(t)\,dt\right]}{\psi(x)} \sim \frac{\dfrac{\psi(x)}{\psi(\lambda x)}\left[\displaystyle\int_0^x \frac{f(t)}{\psi(t)}\psi'(t)\,dt\right]}{\displaystyle\int_0^x \psi'(t)\,dt},$$

$$J = \frac{f(x)}{\psi(x)}\cdot\frac{\psi(x)}{\psi(\lambda x)}\log\frac{\psi(\lambda x)}{\psi(x)}.$$

Since $\psi(\lambda x)/\psi(x) \sim \lambda^c$ by hypothesis, we get

$$\liminf I = \frac{1}{\lambda^c}\liminf \frac{\displaystyle\int_0^x \frac{f(t)}{\psi(t)}\psi'(t)\,dt}{\displaystyle\int_0^x \psi'(t)\,dt} \geqslant \frac{1}{\lambda^c}\liminf\frac{f(x)}{\psi(x)},$$

$$\limsup J = \frac{\log \lambda^c}{\lambda^c}\limsup\frac{f(x)}{\psi(x)}.$$

Hence

$$\limsup\frac{g(\lambda x)}{\psi(\lambda x)} \geqslant \liminf I + \limsup J \geqslant \frac{a}{\lambda^c} + \frac{A\log\lambda^c}{\lambda^c}.$$

This proves (1).

The proof of (2) is similar.

NOTE. $\psi(x)$ in Theorem 1 can be an L-function

$$x^c(\log\ x)^{c_1}(\log\log x)^{c_2}\ldots.$$

It is easily verified that the right-hand member of (1) has an absolute maximum and that of (2) an absolute minimum. The corollary which follows gives the best forms of (1) and (2) in which the right-handed member of (1) is replaced by its maximum and that of (2) by its minimum.

COROLLARY 1. *Under the conditions of Theorem* 1,

$$\limsup_{x\to\infty}\frac{g(x)}{\psi(x)} \geqslant A\,e^{-(A-a)/A}, \tag{3}$$

$$\liminf_{x\to\infty}\frac{g(x)}{\psi(x)} \leqslant a + a\log(A/a). \tag{4}$$

THEOREM 2. *Let $f(z)$ be an entire function of order ρ, $0 < \rho < \infty$, and type τ defined by*

$$\tau = \limsup_{r \to \infty} \frac{\log M(r)}{r^\rho}, \text{ where } M(r) = \max_{|z|=r} |f(z)|.$$

Let

$$\limsup_{r \to \infty} \frac{n(r)}{r^\rho} = L, \quad \liminf_{r \to \infty} \frac{n(r)}{r^\rho} = l,$$

where $n(r)$ is the number of zeroes of f in $|z| \leqslant r$. Then

$$L\, e^{-(L-l)/L} \leqslant \rho\tau. \tag{5}$$

PROOF. Suppose (as we may) that $f(0) \neq 0$. Then, by the definition of τ and Jensen's inequality,

$$\rho\tau = \rho \limsup_{r \to \infty} \frac{\log M(r)}{r^\rho} \geqslant \limsup_{r \to \infty} \frac{1}{r^\rho} \int_0^r n(x) \frac{\rho}{x}\, dx = \limsup_{r \to \infty} \frac{g(r)}{r^\rho},$$

where $g(x)$ is defined exactly as in Theorem 1, with $\phi(x) = \rho/x$ for $x \geqslant 1$, $\phi(x) = 0$ otherwise, i.e. $c = \rho$, $\psi(x) = x^\rho$, $f(x) = n(x)$, i.e. $A = L$, $a = l$.

With these substitutions for ϕ and f, (3) becomes (5).

COROLLARY 2. *(5) can be written in the alternative forms*

$$L \leqslant \rho\, \tau\, e^{(L-l)/L} \leqslant \rho\, \tau\, e, \tag{6}$$

$$l + \sum_{\nu=2}^{\infty} (-1)^\nu \frac{\{(L-l)/L\}^\nu}{\nu\,!} \leqslant \rho\, \tau. \tag{7}$$

(6) shows that, when $L = \rho\tau e$, which is a possibility that can be realized [2, p. 147], *we have $l = 0$. (7) shows that, when $l = \rho\tau$, which is again a possibility that can be realized* [1, p. 32], *we have $L = l$.*

S. M. Shah has proved Theorem 2 directly [3, Lemma 2.1]. Theorem 1 includes, besides Theorem 2, another theorem of S. M. Shah [4, Theorem 1] on entire functions.

I wish to thank Prof. C. T. Rajagopal for his help in preparing this note.

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Madras Christian College
Tambaram

ON RINGS OF ENTIRE FUNCTIONS OF FINITE ORDER

By MELVIN HENRIKSEN

[Received March 26, 1953]

In 1940, Helmer showed [1, Theorem 9] that in the ring R of entire functions, every finitely generated ideal is principal. That is, if f, g are entire functions without zeros in common, there exist s, t in R such that

$$sf + tg = 1. \tag{1}$$

He asked if this theorem is true in the ring R^* of all entire functions of finite order. A negative answer to this question existed already in 1936 in a paper of Whittaker [7, p. 256]. In particular if the zeros (a_n), (b_n) of f, g respectively are not sufficiently separated as $n \to \infty$, the equation (1) cannot hold with s, t in R^*. In 1940, making use of results of [7], Mursi showed [6] that if there is an $h > \max(\text{ord } f, \text{ord } g)$ such that the circles $S(a_n, |a_n|^{-h})$ with center a_n and radius $|a_n|^{-h}$ intersect none of the corresponding circles $S(b_m, |b_m|^{-h})$, then (1) holds with both ord s and ord t no greater than max (ord f, ord g).

In an earlier paper [2], the author showed that if M is any maximal ideal of R, the residue class field R/M is isomorphic with the complex field K. In this paper, under some restrictions, this theorem is extended to the ring R_λ of all entire functions of order no greater than λ, and hence to R^*.

DEFINITION. *Let $i_h(f, n)$ be the number of zeros of f contained in $S(a_n, |a_n|^{-h})$, where a zero of multiplicity m is counted m times.*

THEOREM. *Let M' be a maximal ideal of R_λ containing a function f such that $i_h(f, n)$ is bounded for some h. Then R_λ/M' is isomorphic with K.*

The proof proceeds as in [2, Theorem 6]. It is easily seen that R_λ/M' has degree of transcendency c (where c is the cardinal number of the continuum) over the rational field. By a well-known theorem of Steinitz, it is only necessary to show that R_λ/M' is algebraically closed.

Since a maximal ideal of any integral domain is prime, there is a g in M' such that $i_h(g, n) = 1$, for all n. In particular, all the zeros (b_n) of g are simple. Let $\Phi(z, X) = X^m + f_1(z) X^{m-1} + \dots + f_m(z)$ be a polynomial with coefficients in R_λ/M', of degree $m > 0$. For each n, $\Phi(b_n, X)$ is a polynomial with coefficients in K, which has m complex roots. Choose any such and call it δ_n. It is well known that $|\delta_n| < 1 + \max(|f_1(b_n)|, \dots, |f_m(b_n)|)$. Since the order of the f_i, and the exponent of convergence of (b_n) do not exceed λ, it follows from a theorem of Macintyre and Wilson [5, Theorem 4] (also obtained independently by Leont'ev [4]) that there is a t in R_λ such that $t(b_n) = \delta_n$. So $\Phi(z, t(z))$ is in M', whence the theorem.

REMARKS: 1. The author does not know if there is a maximal ideal in R_λ that fails to satisfy the hypothesis of the theorem.

2. There exist prime ideals of R_λ and R^* that fail to satisfy this hypothesis. For, the set B of elements of f of R_λ (or R^*) with $i_h(f, n)$ bounded for some h, is closed under multiplication. Hence, one can construct, with the aid of Zorn's lemma, prime ideals not intersecting B. See also [3].

3. If, in the theorem R_λ is replaced by R^*, the constant h in the definition of $i_h(f, n)$ can be replaced by a positive, increasing function of $|z|$ such that $\limsup_{n\to\infty} \dfrac{\log h(|a_n|)}{\log |a_n|}$ is finite. See [5, Theorem 5].

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Purdue University
Lafayette, Indiana, U. S. A.

ON A PROBLEM OF HARDY ON THE DISTRIBUTION OF INTEGERS HAVING A GIVEN NUMBER OF PRIME FACTORS, I

By L. G. SATHE

[Received April 4, 1953]

CHAPTER 1

INTRODUCTION

1. Let (i) $\sigma_\nu(x)$ denote the number of numbers $\leqslant x$ which are composed of ν prime factors, where each factor is counted according to its multiplicity, (ii) $\pi_\nu(x)$ denote the number of such square free numbers, (iii) $\rho_\nu(x)$ denote the number of numbers $\leqslant x$ having ν *different* prime factors. Further, let $lx = l_1 x = \log x$; $l_2 x = \log \log x$. It is well known that when ν is finite

$$\sigma_\nu(x) \sim \pi_\nu(x) \sim \rho_\nu(x) \sim \frac{x}{lx} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!}\,. \tag{1}$$

Hardy proposed the problem of determining the asymptotic nature of these functions when ν is a function of x, especially when $\nu = [l_2 x]$. It follows from a Hardy-Ramanujan result that if $\nu \leqslant k l_2 x$, then for some positive $H = H(k)$,

$$\pi_\nu(x) \leqslant \rho_\nu(x) \leqslant H \frac{x}{lx} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!}\,. \tag{2}$$

Pillai and Erdös proved that for some positive H,

$$\rho_\nu(x) > H \frac{x}{lx} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!}, \tag{3}$$

if $$l_2 x - H_1 \sqrt{(l_2 x)} < \nu < l_2 x + H_1 \sqrt{(l_2 x)}.$$

Pillai has proved a good deal more, namely, that if $0 < k < e$ and $\nu \leqslant k l_2$, then

$$\sigma_\nu(x) \geqslant \pi_\nu(x) > H \frac{x}{lx} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!}, \tag{4}$$

and that (1) is true if $\nu = o\,(l_2 x)$. Since $\rho_\nu(x) \geqslant \pi_\nu(x)$, (4) is true

The type-script of this paper was prepared with the kind assistance of Dr. and Mrs. Richard Thomas. *Ed. J. I. M. S.*

for $\rho_\nu(x)$ also. These are the best results proved so far. For the appropriate references see G. H. Hardy's *Ramanujan*, p. 56.

In this paper I prove* that if $\nu \sim kl_2x$, then

$$\pi_\nu(x) \sim f(k)\, \frac{x}{lx} \frac{(l_2x)^{\nu-1}}{(\nu-1)\,!}, \text{ if } k < e\,; \tag{5}$$

$$\sigma_\nu(x) \sim f(k) \left\{ \prod_p \left(1 - \frac{k^2}{p^2}\right)^{-1} \right\} \frac{x}{lx} \frac{(l_2x)^{\nu-1}}{(\nu-1)\,!}, \text{ if } k < 2\,; \tag{6}$$

$$\rho_\nu(x) \sim f(k) \prod_p \left\{1 + \frac{k}{(p+k)\,(p-1)}\right\} \frac{x}{lx} \frac{(l_2x)^{\nu-1}}{(\nu-1)\,!}, \text{ if } k < 2; \tag{7}$$

where

$$f(k) = \frac{1}{\Gamma(k+1)} \left\{ \prod_p \left(1 - \frac{1}{p}\right) e^{1/p} \right\}^k \prod_p \left\{ \left(1 + \frac{k}{p}\right) e^{-k/p} \right\}. \tag{8}$$

In the above products p runs through all the primes. Results (5)-(7) are special cases of the corresponding results for $1 \leqslant \nu \leqslant kl_2x$.

I further deduce from Pillai's result (4) that if $\nu = [2l_2x] + 1$, then

$$\frac{\sigma_\nu(x)\; lx(\nu-1)\,!}{x\;(l_2x)^{\nu-1}} \to \infty \text{ as } x \to \infty. \tag{9}$$

It is easily seen that

$$f(0) = 1, f(1) = 6/\pi^2. \tag{10}$$

Hence, if $\nu \sim l_2x$,

$$\pi_\nu(x) \sim \frac{6}{\pi^2} \frac{x}{lx} \frac{(l_2x)^{\nu-1}}{(\nu-1)\,!}, \tag{11}$$

$$\sigma_\nu(x) \sim \rho_\nu(x) \sim \frac{x}{lx} \frac{(l_2x)^{\nu-1}}{(\nu-1)\,!}. \tag{12}$$

It is very interesting to note that (12) is *not* true when $\nu \sim k\, l_2\, x$, $0 < k < 1$, though it is true when $\nu = o(l_2 x)$ and $\nu \sim l_2\, x$. The behaviour of the constants multiplying $\frac{x}{lx} \frac{(l_2\,x)^{\nu-1}}{(\nu-1)\,!}$ in the formulae (5)-(7) is very remarkable. Thus, for instance, $f(k)$ is maximum when $k = \cdot 14\ldots$, while $f(k)\{\prod_p (1 - k^2/p^2)^{-1}\}$ has both maximum and minimum values in the range $0 \leqslant k < 2$. We can

* Cf. The *Math. Student*, 13 (1945), p. 153 and *Annals of Math.* 49 (1948), pp. 54-56.

deduce (6) and (7) as consequences of (5), and also prove them quite independently. The results by the two methods are in perfect agreement, except that we get *sharper estimates of the error by* direct proofs.

2. Notations. q_ν denotes a square free number having ν prime factors. For r an integer $\geqslant 2$ we define

$$\lambda(r) = \sum_p 1/p^r, \quad \zeta(r) = \sum_{n=1}^{\infty} 1/n^r,$$
$$A(r) = \lambda(r) + \zeta(r).$$

p denotes a prime number throughout.

$$B_0 = 1,\ B = B_1 = \lim_{x=\infty} \Big\{ \sum_{p \leqslant x} 1/p - l_2 x \Big\}.$$

For $r \geqslant 2$, B_r is defined by the recurrence relation

$$r\,B_r = B_{r-1}\,B + \sum_{t=2}^{r} (-1)^{t-1}\,B_{r-t}\,A(t),$$

$$\psi(\nu, x) = \frac{x}{lx} \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x)^{\nu-r-1}}{(\nu - r - 1)\,!}, \tag{13}$$

$$R_\nu(x) = \pi_\nu(x) - \psi(\nu, x). \tag{14}$$

$a_\nu = 2.3 \ldots p_\nu$ denotes the smallest square free number having ν prime factors.

$c, c', c_1, \ldots ; b, b_1, b_2, \ldots$ are *absolute* positive constants.

$$c' \geqslant \max \{ 1, c_1, c_2, \ldots, c_{72} ; b, b_1, \ldots, b_{122} \}. \tag{16}$$

$$h(\nu) = \prod_{m=1}^{\nu} \left(1 + \frac{c'}{\sqrt{m}} \right). \tag{17}$$

$\alpha = \frac{1}{2}$. We write $f_1(x) = \theta(f_2(x))$, if $|f_1(x)| \leqslant f_2(x)$. γ is Euler's constant.

3. The method. From the classical results it follows, in particular, that when $l_2\,x \geqslant 1$,

$$\pi_1(x) = \pi(x) = \frac{x}{lx} + \theta \left\{ c_1 \frac{x}{(lx)^2} l_2\,x \right\}. \tag{18}$$

Also as a consequence of Landau's result we have

$$\pi_2(x) = \frac{x}{lx} (l_2 x + B) + \theta \left\{ c_2 \frac{x}{(lx)^2} \frac{(l_2\,x)^2}{2\,!} \right\}, \tag{19}$$

when $l_2 x \geqslant 2$. We shall prove generally that when $l_2 x \geqslant \nu$,

$$\pi_\nu(x) = \psi(\nu, x) + \theta \left\{ c \frac{x}{(lx)^2} \frac{(l_2 x)^\nu}{\nu!} h(\nu) \right\}, \tag{20}$$

where c may be any constant $\geqslant c_1 + c_2$. (20) is obviously equivalent to :

If $l_2 x \geqslant \nu$, then

$$|R_\nu(x)| \leqslant c \frac{x}{(lx)^2} \frac{(l_2 x)^\nu}{\nu!} h(\nu). \tag{21}$$

The proof of (21) requires a very elaborate analysis and occupies almost the whole of the paper. The deduction of the asymptotic formula from (21) for $\nu \leqslant l_2 x$ and generalization to the case $\nu \leqslant k\, l_2 x$, where $k < e$ is then a simple affair.

By virtue of (18) and (19), (21) is true for $\nu = 1, 2$, since $h(\nu) \geqslant 1$. Let now ν' be any integer not less than 2. Assuming that

$$|R_\nu(x)| \leqslant c \frac{x}{(lx)^2} \frac{(l_2 x)^\nu}{\nu!} h(\nu),$$

when $l_2 x \geqslant \nu$, for every $\nu \leqslant \nu'$ we show (after a very lengthy analysis) that

$$|R_{\nu'+1}(x)| \leqslant c \frac{x}{(lx)^2} \frac{(l_2 x)^{\nu'+1}}{(\nu'+1)!} h(\nu'+1), \tag{22}$$

when $l_2 x \geqslant \nu' + 1$. Thus it will follow that if (21) is true for $\nu \leqslant \nu'$ then it is true for $\nu' + 1$. Hence (21) is proved by induction by virtue of (18) and (19) for *all* ν.

Hereafter, up to the end of Chapter 6 *we assume that* (21) *is true for* $1 \leqslant \nu \leqslant \nu'$, *where we suppose that* ν' *is any fixed integer* $\geqslant 2$. (23)

Pillai has proved that

$$(\nu + 1)\, \pi_{\nu+1}(x) = \sum_{r=0}^{\nu} (-1)^r \sum_{p^{r+1} \leqslant x} \pi_{\nu-r}(x/p^{r+1}), \tag{24}$$

where $\pi_0(x) = 1$ if $x \geqslant 1$ and equal to 0 if $x < 1$. If now $\nu \leqslant \nu'$, then the right side of (24) can be evaluated with the help of (23). We thus estimate $\pi_{\nu+1}(x)$, and hence $R_{\nu+1}(x)$ for $\nu \leqslant \nu'$. For $\nu = \nu'$ this gives the results which we have already assumed.

The details, however, appear to be sufficiently complicated. The sum

$$\sum_{r=1}^{\nu} (-1)^r \sum_{p^{r+1}\leqslant x} \pi_{\nu-r}(x/p^{r+1}),$$

on the right side of (24) can be disposed of easily. The sum $\sum_{p\leqslant x} \pi_\nu(x/p)$, however, presents very serious difficulties.

4. Evalution of $\sum_{p\leqslant x} \pi_\nu(x/p)$ for $\nu \leqslant \nu'$. The result for this sum (under the assumption (23)) when $l_2 x \geqslant \nu+1$ is given in Lemma 44. This is obtained by using the transformation formula of Pillai

$$\sum_{p\leqslant x} \pi_\nu(x/p) = \sum_{p\leqslant x^{1-a}} \pi_\nu(x/p) + \sum_{q_\nu\leqslant x^a} \pi(x/q_\nu) - \pi(x^{1-a})\,\pi_\nu(x^a); \qquad (25)$$

(25), in fact, is true for any a satisfying $0 < a < 1$.

Since $l_2\,(x/p) \geqslant \nu$ for $l_2\,x \geqslant \nu+1$, $p \leqslant x^{1-a}$, we can evaluate, after easy calculations, the sum $\sum_{p\leqslant x^{1-a}} \pi_\nu(x/p)$ for $\nu \leqslant \nu'$, $l_2 x \geqslant \nu+1$ under assumption (23). The dominant term in

$$\sum_{q_\nu\leqslant x^a} \pi(x/q_\nu),$$

is $\sum_{q_\nu\leqslant x^a} x/q_\nu\,(lx - lq_\nu)$. We evaluate it by estimating

$$\sum_{q_\nu\leqslant x^a} (1/q_\nu), \text{ for } \nu \leqslant \nu',\ l_2 x \geqslant \nu+1, \qquad (26)$$

and

$$\sum_{q_\nu\leqslant x^a} (\log q_\nu)^n/q_\nu, \text{ for } 1 \leqslant n \leqslant 2\,l_2 x,\ \nu \leqslant \nu',\ l_2 x \geqslant \nu+1. \qquad (27)$$

From (26) and (27) we can easily estimate $\sum_{q_\nu\leqslant x^a} \pi(x/q_\nu)$ and hence $\sum_{p\leqslant x} \pi_\nu(x/p)$ is finally evaluated with the necessary approximation to the error term.

5. The general scheme of the paper. Chapter 3 is devoted to estimate (26) and Chapter 4 for (27). In Chapter 5 we consider the sum $\sum_{p\leqslant x^{1-a}} \pi_\nu(x/p)$. In Chapter 6, (22) is proved under the assumption that (21) holds for $\nu \leqslant \nu'$. Hence (21) follows by induction for *all* ν.

Chapter 7 is devoted to prove the main results (5)-(7). Here (6) and (7) are deduced directly from the results for $\pi_\nu(x)$. Chapter 8 deals with an independent proof for $\sigma_\nu(x)$ and $\rho_\nu(x)$ without any reference to $\pi_\nu(x)$. The proof is based on precisely the same principles, starting from the recurrence relations for $\sigma_\nu(x)$ and $\rho_\nu(x)$ instead of (24). We have only to modify our previous arguments trivially. The results of Chapters 7 and 8 are in complete harmony.

In Chapter 9 equation (9) and other miscellaneous results are proved. At the end, a table is supplied giving the variations in the functions of k multiplying $\dfrac{x}{\log x}\dfrac{(l_2x)^{\nu-1}}{(\nu-1)\,!}$ in the formulae (5)-(7).

Before we proceed with the proof it is necessary to know the behaviour of $|B_r|$ as $r\to\infty$ and the sum of the series $\sum_{r=0}^{\infty} B_r z^r$. This is done in Chapter 2.

CHAPTER 2

On $|B_s|$

Lemma 1. *If*

$$f(z)=\frac{1}{\Gamma(z+1)}\prod_p\{(1-1/p)\,e^{1/p}\}^z\prod_p\{(1+z/p)\,e^{-z/p}\},\tag{28}$$

then, for all finite values of z,

$$f(z)=\sum_{r=0}^{\infty}B_r z^r,\tag{29}$$

where B_r is defined as in (13).

Proof. By definition, $f(z)$ is a product of integral functions and hence an integral function of z. It is therefore sufficient to prove (19) for $|z|\leqslant\frac{1}{2}$.

The series $Bz-\dfrac{A(2)}{2}z^2+\dfrac{A(3)}{3}z^3-\dots$ has $|z|=1$ as its circle of convergence, since $|A(n)-1|\leqslant 4/2^n$ for $n\geqslant 4$. We shall

show, in the first instance, that for $|z| \leqslant \frac{1}{2}$,

$$f(z) = e^{Bz - \frac{A(2)}{2} z^2 + \frac{A(3)}{3} z^3 - \dots} \tag{30}$$

By definition,

$$- \frac{A(2)}{2} z^2 + \frac{A(3)}{3} z^3 - \dots$$

$$= \left\{ - \frac{z^2}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{z^3}{3} \sum_{n=1}^{\infty} \frac{1}{n^3} - \dots \right\} + \left\{ - \frac{z^2}{2} \sum_{p} \frac{1}{p^2} + \frac{z^3}{3} \sum_{p} \frac{1}{p^3} - \dots \right\}$$

$$= \sum_{n=1}^{\infty} \left(- \frac{z^2}{2n^2} + \frac{1}{3} \frac{z^3}{n^3} - \dots \right) + \sum_{p} \left(- \frac{1}{2} \frac{z^2}{p^2} + \frac{1}{3} \frac{z^3}{p^3} - \dots \right)$$

$$= \log \prod_{n=1}^{\infty} \left\{ (1 + z/n)\, e^{-z/n} \right\} + \log \prod_{p} \left\{ (1 + z/p)\, e^{-z/p} \right\}, \tag{31}$$

the inversion of the order of summation being valid as the double series

$$\sum_{\substack{m,\, n \\ m > 1}} \frac{1}{mn^m} |z|^m < \sum_{\substack{m,\, n \\ m > 1}} \frac{1}{m 2^m n^m} \leqslant c_3 \sum_{m,\, n} \frac{1}{m^2 n^2}$$

is convergent. It is understood that in (31) principal values of the logarithms are taken.

Also [cf. A. E. Ingnam : *The distribution of prime numbers,* Cambridge Tract, 30 (1932), p. 23]

$$B = \gamma + \log \left\{ \Pi (1 - 1/p)\, e^{1/p} \right\}.$$

Hence

$$e^{Bz} = e^{\gamma z} \prod_{p} \left\{ (1 - 1/p)\, e^{1/p} \right\}^z. \tag{32}$$

From (31) and (32) the right side of (30) is equal to

$$e^{\gamma z} \prod_{n=1}^{\infty} \left\{ (1 + z/n)\, e^{-z/n} \right\} \prod_{p} \left\{ (1 + z/p)\, e^{-z/p} \right\} \prod_{p} \left\{ (1 - 1/p)\, e^{1/p} \right\}^z = f(z).$$

The right side of (30) represents an analytic function of z for $|z| \leqslant \frac{1}{2}$. Hence

$$f(z) = e^{Bz - \frac{A(2)}{2} z^2 + \dots} = \sum_{r=0}^{\infty} a_r z^r, \text{ say.} \tag{33}$$

By actual expansion we get $a_0 = 1$, $a_1 = B$. Hence to prove (29) it is sufficient to show that for $r \geqslant 2$, $a_r = B_r$; or that a_r satisfies

the same recurrence relation as B_r for $r > 2$. Differentiating (33) we get

$$\left(\sum_{r=0}^{\infty} a_r z^r\right)\{B - A(2)z + A(3)z^2 - \ldots\} = \sum_{r=1}^{\infty} r\, a_r z^{r-1}. \quad (34)$$

Equating the coefficients of z^{r-1} on both sides of (34) we get

$$r\, a_r = a_{r-1}B + \sum_{t=2}^{r} (-1)^{t-1} a_{r-t} A(t),$$

and this is the same as (13) with a's in place of B's.

LEMMA 2. $|B_r| \leqslant c_4/(10)^{4r}$, *for* $r \geqslant 0$. (35)

PROOF. $\sum_{r=0} B_r z^r$ represents an integral function of z by virtue of Lemma 1.

CHAPTER 3

ON THE SUM $\sum_{q_\nu \leqslant x^\alpha} 1/q_\nu$

INTRODUCTION. Hereafter, up to the end of Chapter 6, it will be understood that $\nu' \geqslant 2$ is the fixed integer determined by (23). Further we define

$$S_\nu(x) = \sum_{q_\nu \leqslant x} 1/q_\nu$$

$$E_\nu(x) = S_\nu(x) - \sum_{r=0}^{\nu} B_r \frac{(l_2 x)^{\nu-r}}{(\nu - r)!}. \quad (36)$$

The final result proved in this chapter is :

If (23) *holds*, $1 \leqslant \nu \leqslant \nu'$, *then, when* $l_2 x \leqslant \nu$,

$$|E_\nu(x)| \leqslant c_5 \frac{(l_2 x)^\nu}{\nu!} \frac{\sqrt{\nu}\, h(\nu)}{\log x}. \quad (37)$$

It follows, in particular, from (37) that for $1 \leqslant \nu \leqslant \nu'$

$$S_\nu(x^\alpha) = \sum_{r=0}^{\nu} B_r \frac{(l_2 x^\alpha)^{\nu-r}}{(\nu - r)!} + \theta\left\{c_6 \frac{(l_2 x)^\nu}{\nu!} \frac{\sqrt{\nu}\, h(\nu)}{\log x}\right\}, \quad (38)$$

when $l_2 x \geqslant \nu + 1$. This will complete the investigation of (26).

THE METHOD. We have, if $x \geqslant 2$,

$$\sum_{q_\nu \leqslant x} \frac{1}{q_\nu} = \frac{\pi_\nu(x)}{x} + \int_{a_\nu}^{x} \frac{\pi_\nu(u)}{u^2}\,du$$

$$= \frac{\pi_\nu(x)}{x} + \int_{a_\nu}^{x} \frac{\psi(\nu, u)}{u^2}\,du + \int_{a_\nu}^{\infty} \frac{R_\nu(u)}{u^2}\,du - \int_{x}^{\infty} \frac{R_\nu(u)}{u^2}\,du, \quad (39)$$

by virtue of (14).

We shall see later that the integral $\int_{a_\nu}^{\infty} \frac{R_\nu(u)}{u^2}\,du$ is convergent. It will enter in (36) as a part of the dominant term and hence it must be evaluated in terms of known quantities. This will be the main task. The value of the integral will be shown to be equal to

$\sum_{r=0}^{\nu} B_r \frac{(l_2\, a_\iota)^{\nu-r}}{(\nu - r)\,!}$ if $1 \leqslant \nu \leqslant \nu'$.

The principle underlying the method of evaluating this integral is essentially the one illustrated in Ingham's tract [*loc. cit.*]. We equate the values of the expression $\sum_{q_\nu} 1/q_\nu^{1+\delta}$ found out in two ways, namely those given by the equations (77) and (78), one of them involving the integral $J(\nu, \delta) = \int_{a_\nu}^{\infty} \frac{R_\nu(u)}{u^{2+\delta}}\,du$. Finally making $\delta \to 0$ in the equation so obtained, we get the value of

$$\lim_{\delta \to 0} \int_{a_\nu}^{\infty} \frac{R_\nu(u)}{u^{2+\delta}}\,du. \quad (40)$$

The passage from (40) to $\int_{a_\nu}^{\infty} \frac{R_\nu(u)du}{u^2}$ presents no difficulty as it can be easily shown that the integral in (40) is uniformly convergent for $\delta \geqslant 0$.

Throughout this chapter we shall denote by $\phi(\nu)$ a function depending on ν but *not on* δ. *The same symbol* $\phi(\nu)$ *may denote different functions on different occasions.* Further we denote

$$s = 1 + \delta; \qquad 0 < \delta \leqslant 1/20, \quad (41)$$

$$F_\nu(s) = \sum_{q_\nu} 1/q_\nu^s, \quad (42)$$

$$F_0(s) = 1.$$

It may be mentioned explicitly that the constants c are absolute positive constants and are independent of δ.

LEMMA 3. *If $\nu \leqslant l_2 x$, then*

$$\frac{(l_2x)^\nu}{\nu\,!} + \frac{(l_2x)^{\nu-1}}{(\nu-1)\,!} + \ldots + \frac{1}{1\,!} = \theta\left\{c_7\frac{(l_2x)^\nu}{\nu\,!}\sqrt{\nu}\right\}.$$

PROOF.

$$\begin{aligned}
&\frac{(l_2x)^{\nu-1}}{(\nu-1)\,!} + \frac{(l_2x)^{\nu-2}}{(\nu-2)\,!} + \ldots + 1 \\
&= \frac{(l_2x)^{\nu-1}}{(\nu-1)\,!}\left\{1 + \frac{(\nu-1)}{l_2x} + \ldots + \frac{(\nu-1)\,!}{(l_2x)^{\nu-1}}\right\} \\
&= \frac{(l_2x)^{\nu-1}}{(\nu-1)\,!}\int_0^\infty e^{-z}\left(1+\frac{z}{l_2x}\right)^{\nu-1} dz. \qquad (43)
\end{aligned}$$

Also

$$\begin{aligned}
\int_0^\infty e^{-z}\left(1+\frac{z}{l_2x}\right)^{\nu-1} dz &\leqslant \int_0^\infty e^{-z}\left(1+\frac{z}{l_2x}\right)^{l_2x} dz \\
&\sim \sqrt{(\pi/2)l_2x} \\
&\leqslant c_8\sqrt{l_2x} \leqslant c_8\frac{l_2x}{\sqrt{2}}.
\end{aligned}$$

Hence the given expression does not exceed

$$\frac{(l_2x)^\nu}{\nu!} + c_8\frac{(l_2x)^\nu}{\nu\,!}\sqrt{\nu} = \frac{(l_2x)^\nu}{\nu!}\sqrt{\nu}\left\{\frac{1}{\sqrt{\nu}} + c_8\right\}.$$

Taking $c_7 \geqslant 1 + c_8$ the lemma follows.

LEMMA 4. *For $\nu \geqslant 2$,*

$$\begin{aligned}
\nu F_\nu(s) = (\Sigma\, 1/p^s)\, F_{\nu-1}(s) - (\Sigma\, 1/p^{2s})\, F_{\nu-2}(s) + \ldots \\
\ldots + (-1)^{t-1}(\Sigma\, 1/p^{ts})\, F_{\nu-t}(s) + \ldots \\
\ldots + (-1)^{\nu-2}(\Sigma\, 1/p^{(\nu-1)s})\, F_1(s) + (-1)^{\nu-1}\Sigma\, 1/p^{\nu s}. \qquad (44)
\end{aligned}$$

PROOF.

$$\begin{aligned}
\nu F_\nu(s) &= \nu \sum_{q_\nu} 1/q_\nu^s \\
&= \sum_p 1/p^s \sum_{\substack{q_{\nu-1} \\ (p,q_{\nu-1})=1}} 1/q^s_{\nu-1} \\
&= (\Sigma\, 1/p^s)\sum_{q_{\nu-1}} 1/q^s_{\nu-1} - \sum_p 1/p^{2s} \sum_{\substack{q_{\nu-2} \\ (p,q_{\nu-2})=1}} 1/q^s_{\nu-2} \\
&= (\Sigma\, 1/p^s)\, F_{\nu-1}(s) - (\Sigma\, 1/p^{2s})\, F_{\nu-2}(s) + \\
&\qquad + \Sigma\, 1/p^{3s} \sum_{\substack{q_{\nu-3} \\ (p,q_{\nu-3})=1}} 1/q^s_{\nu-3}.
\end{aligned}$$

Proceeding thus the result follows.

LEMMA 5.

$$F_1(s) = \Sigma\, 1/p^s = -\log\delta + B + \Gamma' + \theta\,(c_9\,\delta),$$

where

$$\Gamma' = \left\{\frac{d}{dx}\,\Gamma(x)\right\}_{x=1} = -\gamma. \tag{45}$$

PROOF.

$$\begin{aligned}\log\zeta(s) &= -\sum_p \log(1 - 1/p^s)\\ &= \Sigma\, 1/p^s + \sum_p \sum_{m=2}^{\infty} 1/mp^{ms}\\ &= \Sigma\, 1/p^s\, \sum_p \sum_{m=2}^{\infty} 1/mp^m + \sum_p \sum_{m=2}^{\infty} (1/m)\left\{1/p^{m(1+\delta)} - 1/p^m\right\}.\end{aligned} \tag{46}$$

Now

$$-\, m\log p \int_1^{1+\delta} \frac{du}{p^{mu}} = \frac{1}{p^{m(1+\delta)}} - \frac{1}{p^m},$$

$$\left|\frac{1}{p^{m(1+\delta)}} - \frac{1}{p^m}\right| \leqslant m\,\frac{\log p}{p^m}\int_1^{1+\delta} du = m\,\frac{\log p}{p^m}\,.\,\delta. \tag{47}$$

Hence

$$\begin{aligned}\sum_p \sum_{m=2}^{\infty}\left|\frac{1}{m}\left(\frac{1}{p^{m(1+\delta)}} - \frac{1}{p^m}\right)\right| &< \delta \sum_p \sum_{m=2}^{\infty} \frac{\log p}{p^m}\\ &= \delta \sum_p \frac{\log p}{p(p-1)}\\ &= c_{10}.\,\delta.\end{aligned} \tag{48}$$

Also

$$\log\zeta(s) = -\log\delta + \theta\,(c_{11}\,\delta).$$

Hence from (46) and (48)

$$\begin{aligned}F_1(s) &= -\log\delta - \sum_p \sum_{m=2}^{\infty} 1/mp^m + \theta\left\{(c_{10} + c_{11})\,\delta\right\}\\ &= -\log\delta + B + \Gamma' + \theta\left\{(c_{10} + c_{11})\,\delta\right\},\end{aligned}$$

since $B = \gamma + \sum_p \sum_{m=2}^{\infty} 1/mp^m = -\Gamma' + \sum_p \sum_{m=2}^{\infty} 1/mp^m.$

Taking $c_9 \geqslant c_{10} + c_{11}$ the lemma follows.

LEMMA 6. *For every integer* $\nu \geqslant 1$,

$$\nu F_\nu(s) = (-\log\delta + B + \Gamma')\, F_{\nu-1}(s) - \\ - \lambda(2)\, F_{\nu-2}(s) + \ldots + (-1)^{t-1}\lambda(t)\, F_{\nu-t}(s) + \ldots + \\ + (-1)^{\nu-1}\lambda(\nu) + \theta\,\{\phi(\nu)\,|\log\delta|^{\nu-1}\,\delta\}.$$

PROOF. We shall show, in the first instance, that

$$|F_\nu(s)| \leqslant \phi(\nu)\,|\log\delta|^\nu. \tag{49}$$

(49) is evidently true when $\nu = 1$ by virtue or Lemma 5. If now (49) is true for $F_1(s), \ldots, F_{\nu-1}(s)$, then by Lemmas 4 and 5

$$|\nu F_\nu(s)| \leqslant \phi(\nu)\,|\log\delta|\,|\log\delta|^{\nu-1} + (\Sigma\, 1/p^2)\,\phi(\nu).\,|\log\delta|^{\nu-2} + \ldots \\ \leqslant \phi(\nu).\,|\log\delta|^\nu,$$

$\phi(\nu)$ being independent of δ. From this, (49) follows by induction. Now, for $2 \leqslant t \leqslant \nu$ we get by virtue of (47)

$$\sum_p 1/p^{t(1+\delta)} = \Sigma\, 1/p^t + \Sigma\,(1/p^{t(1+\delta)} - 1/p^t) \\ = \lambda(t) + \theta(\nu\,\delta\,\Sigma\log p/p^2) \\ = \lambda(t) + \theta(c_{12}\,\nu\,\delta). \tag{50}$$

Hence by Lemmas 4, 5 and equation (50) we get

$$\nu F_\nu(s) = \{-\log\delta + B + \Gamma' + \theta(c_9\,\delta)\}\, F_{\nu-1}(s) - \\ - \lambda(2)\, F_{\nu-2}(s) + \theta\,\{c_{12}\,\nu\,\delta\, F_{\nu-2}(s)\} + \ldots + \\ + (-1)^{\nu-1}\lambda(\nu) + \theta(c_{12}\,\nu\,\delta) \\ = (-\log\delta + B + \Gamma')\, F_{\nu-1}(s) - \lambda(2)\, F_{\nu-2}(s) + \ldots + \\ + (-1)^{\nu-1}\lambda(\nu) + \theta\,\{\delta(c_9\,|F_{\nu-1}(s)| + \nu c_{12}\,|F_{\nu-2}(s) + \\ + \ldots \text{ to } \nu \text{ terms})\}.$$

The lemma now follows on using (49).

LEMMA 7. *Let*

$$\Gamma^{(n)} = \Gamma^{(n)}(1) = \left\{\frac{d^n}{dx^n}\Gamma(x)\right\}_{x=1} \tag{51}$$

Then for every integer $n \geqslant 1$

$$\Gamma^{(n)} = \int_0^\infty e^{-w}\,(\log w)^n\, dw, \tag{52}$$

and

$$\frac{\Gamma^{(n)}}{(n-1)\,!} = \frac{\Gamma^{(n-1)}}{(n-1)\,!}\Gamma' + \sum_{t=1}^{n-1}(-1)^{t+1}\,\zeta(t+1)\,\frac{\Gamma^{(n-t-1)}}{(n-t-1)\,!}, \tag{53}$$

the empty sum when $n = 1$ *being regarded as equal to* 0.

PROOF. (52) is obtained by differentiating

$$\Gamma(z) = \int_0^\infty e^{-x}\, x^{z-1}\, dx,$$

with respect to z, n times and then putting $z = 1$. To prove (53) let $y = \log \Gamma(x)$, so that

$$\Gamma'(x) = \frac{dy}{dx} \,.\, \Gamma(x)\,. \tag{54}$$

Differentiating (54) $(n-1)$ times, and putting $x = 1$ and using the result

$$\left\{ \frac{d^n}{dx^n} \log \Gamma(x) \right\}_{x=1} = (-1)^n (n-1)!\, \zeta(n),$$

we get (53).

LEMMA 8. *For* $\mu \geqslant t \geqslant 0$,

$$(B + \Gamma') \sum_{r=0}^{\mu-t} B_r \frac{\Gamma^{(\mu-t-r)}}{(\mu-t-r)!} + \sum_{t'=1}^{\mu-t} (-1)^{t'} \lambda(t'+1) \times$$
$$\times \sum_{r=0}^{\mu-t-t'} B_r \frac{\Gamma^{(\mu-t-t'-r)}}{(\mu-t-t'-r)!} = (\mu+1-t) \sum_{r=0}^{\mu-t+1} B_r \frac{\Gamma^{(\mu-t-r+1)}}{(\mu-t-r-1)}, \tag{55}$$

the empty sum when $\mu = t$ *being considered equal to* 0.

PROOF. The right side of (55) is equal to

$$(\mu+1-t)\, B_{\mu-t+1} + \sum_{r=0}^{\mu-t} B_r \frac{\Gamma^{(\mu-t-r+1)}}{(\mu-t-r+1)\;!} (\mu+1-t)$$
$$= (\mu+1-t)\, B_{\mu-t+1} + \sum_{r=0}^{\mu-t} B_r \frac{\Gamma^{(\mu-t-r+1)}}{(\mu-t-r)!} +$$
$$+ \sum_{r=0}^{\mu-t} r\, B_r \frac{\Gamma^{(\mu-t-r+1)}}{(\mu-t+r+1)\,!}$$
$$= \sum_{r=0}^{\mu-t} B_r \frac{\Gamma^{(\mu-t-r+1)}}{(\mu-t-r)!} + \sum_{r=0}^{\mu-t+1} r\, B_r \frac{\Gamma^{(\mu-t-r+1)}}{(\mu-t-r+1)!}$$
$$= \Sigma_1 + \Sigma_2, \text{ say.} \tag{56}$$

$$\Sigma_1 = \sum_{r=0}^{\mu-t} B_r \left\{ \frac{\Gamma^{(\mu-t-r)}}{(\mu-t-r)!} \Gamma' + \right.$$
$$\left. + \sum_{t'=1}^{\mu-t-r} (-1)^{t'+1} \zeta(t'+1) \frac{\Gamma^{(\mu-t-r-t')}}{(\mu-t-r-t')!} \right\},$$

by Lemma 7. Hence on changing the order of summation with respect to r and t we get

$$\Sigma_1 = \Gamma' \sum_{r=0}^{\mu-t} B_r \frac{\Gamma^{(\mu-t-r)}}{(\mu-t-r)\,!} + \sum_{t'=1}^{\mu-t} (-1)^{t'+1} \zeta(t'+1) \times$$
$$\times \sum_{r=0}^{\mu-t-t'} B_r \frac{\Gamma^{(\mu-t-r-t')}}{(\mu-t-r-t')\,!}. \qquad (57)$$

Also

$$\Sigma_2 = \sum_{r=1}^{\mu-t+1} r\, B_r \frac{\Gamma^{(\mu-t-r+1)}}{(\mu-t-r+1)\,!}$$
$$= \sum_{r=1}^{\mu-t+1} \left\{ B\, B_{r-1} + \sum_{t'=1}^{r-1} (-1)^{t'} A(t'+1)\, B_{r-t'-1} \right\} \frac{\Gamma^{(\mu-t-r+1)}}{(\mu-t-r+1)!}$$
$$= B \sum_{r=0}^{\mu-t} B_r \frac{\Gamma^{(\mu-t-r)}}{(\mu-t-r)\,!} + \sum_{t'=1}^{\mu-t} (-1)^{t'} A(t'+1) \times$$
$$\times \sum_{r=0}^{\mu-t-t'} B_r \frac{\Gamma^{(\mu-t-r-t')}}{(\mu-t-r-t')!}. \qquad (58)$$

Since

$$(-1)^{t'} A(t'+1) + (-1)^{t'+1} \zeta(t'+1) = (-1)^{t'} \lambda(t'+1),$$

the result follows immediately from (56)-(58).

Lemma 9. *For every integer* $\mu \geqslant 1$

$$F_\mu(s) = \sum_{t=0}^{\mu} (-1)^t \frac{(\log \delta)^t}{t\,!} \sum_{r=0}^{\mu-t} B_r \frac{\Gamma^{(\mu-t-r)}}{(\mu-t-r)\,!} +$$
$$+ \theta \left\{ \phi(u) |\log \delta|^{\mu-1} \delta \right\}, \qquad (59)$$

where $\phi(u)$ *is independent of* δ.

Proof. When $\mu = 1$, (59) reduces to

$$F_1(s) = -(\log \delta)\, B_0 + B_0 \Gamma' + B_1 + \theta\, (c_{13}\, \delta)$$
$$= -\log \delta + \Gamma' + B + \theta\, (c_{13}\, \delta),$$

which is true by virtue of Lemma 5 if $c_{13} \geqslant c_9$.

Assuming now that (59) is true for $F_1(s), \ldots, F_\mu(s)$, $\mu \geqslant 1$ we shall show that it is true for $F_{\mu+1}(s)$ also. By our assumption and

Lemma 6 the coefficient of $(\log \delta)^t$ in the dominant expression for $F_{\mu+1}(s)$ is equal to

$$\frac{(-1)^t}{(t-1)!}\cdot\frac{1}{(\mu+1)}\sum_{r=0}^{\mu-t+1} B_r \frac{\Gamma^{(\mu-t-r+1)}}{(\mu-t-r+1)!}+$$

$$+(B+\Gamma')\frac{(-1)^{t'}}{t'}\frac{1}{(\mu+1)}\sum_{r=0}^{\mu-t} B_r \frac{\Gamma^{(\mu-t-r)}}{(\mu-t-r)!}+$$

$$+\frac{(-1)^t}{t!}\frac{1}{(\mu+1)}\sum_{t'=1}^{\mu-t}(-1)^{t'}\lambda(t'+1)\sum_{r=0}^{\mu-t-t'} B_r \frac{\Gamma^{(\mu-t-t'-r)}}{(\mu-t-t'-r)!}. \tag{60}$$

(60) is true for $1 \leqslant t \leqslant \mu$ and since $\dfrac{1}{(t-1)!}=0$ when $t=0$, it is also true for $t=0$. When $t=\mu+1$ (60) holds with the convention

$$\sum_u^v = 0, \text{ if } v < u. \tag{61}$$

Using Lemma 8 the coefficient of $(\log \delta)^t$ becomes equal to

$$\left\{\frac{(-1)^t}{(t-1)!}\frac{1}{(\mu+1)}+\frac{(-1)^t}{t!}\frac{(\mu+1-t)}{(\mu+1)}\right\}\sum_{r=0}^{\mu-t+1} B_r \frac{\Gamma^{(\mu-t-r+1)}}{(\mu-t-r+1)!}$$

$$=\frac{(-1)^t}{t!}\sum_{r=0}^{\mu-t+1} B_r \frac{\Gamma^{(\mu-t-r+1)}}{(\mu-t-r+1)!}, \tag{62}$$

(62) being valid for $0 \leqslant t \leqslant \mu+1$. Also the error term is equal to

$$\theta\{\phi(\mu)|\log\delta|^\mu\delta+\phi(\mu)|\log\delta|^{\mu-2}\delta+\ldots\}=\theta\{\phi(\mu)|\log\delta|^\mu\delta\}. \tag{63}$$

Hence from (62) and (63) we get

$$F_{\mu+1}(s)=\sum_{t=0}^{\mu+1}\frac{(-1)^t}{t!}(\log\delta)^t\sum_{r=0}^{\mu+1-t} B_r \frac{\Gamma^{(\mu+1-t-r)}}{(\mu+1-t-r)!}+$$
$$+\theta\{\phi(\mu)|\log\delta|^\mu\delta)\}.$$

This is obtained from (59) by writing $\mu+1$ for μ. Hence the lemma follows by induction.

LEMMA 10. *For every integer $n \geqslant 0$ and $a \geqslant 2$,*

$$\int_a^\infty \frac{(l_2u)^n}{n!}\frac{1}{u\log u'}\frac{du}{u^\delta}=\sum_{t=0}^{n+1}\frac{(-1)^t}{t!}\frac{\Gamma^{(n+1-t)}}{(n+1-t)!}(\log\delta)^t-\frac{(l_2a)^{n+1}}{(n+1)!}+$$
$$+\theta\{\phi(n,a)|\log\delta|^{n+1}\delta\}, \tag{64}$$

where $\phi(n, a)$ is independent of δ.

As before the same symbol $\phi(n, a)$ may denote different expressions on different occasions.

PROOF.

$$\int_a^\infty \frac{(l_2 u)^n}{n\,!}\,\frac{1}{u \log u}\,\frac{du}{u^\delta} = \int_{\log a}^\infty \frac{(\log v)^n}{n\,!}\,\frac{e^{-v\delta}}{v}\,dv$$

$$= \frac{1}{n\,!}\int_{\delta \log a}^\infty \frac{(\log w - \log \delta)^n}{w}\,e^{-w}\,dw$$

$$= \frac{1}{(n+1)\,!}\Big[(\log w - \log \delta)^{n+1}\,e^{-w}\Big]_{\delta \log a}^{\infty} +$$

$$+ \frac{1}{(n+1)\,!}\int_{\delta \log a}^\infty (\log w - \log \delta)^{n+1}\,e^{-w}\,dw. \quad (65)$$

Now

$$e^{-\delta \log a} = 1 + \theta\left\{\delta\left(\frac{(\log a)}{1\,!} + \frac{(\log a)^2}{2\,!} + \dots\right)\right\}$$

$$= 1 + \theta(\delta a).$$

Hence the first term on the right side of (65) is equal to

$$-\frac{(l_2\,a)^{n+1}}{(n+1)\,!}\left\{1 + \theta\,(\delta a)\right\}. \quad (66)$$

Also

$$\frac{1}{(n+1)\,!}\int_{\delta \log a}^\infty (\log w - \log \delta)^{n+1}\,e^{-w}\,dw$$

$$= \frac{1}{(n+1)\,!}\int_0^\infty (\log w - \log \delta)^{n+1}\,e^{-w}\,dw -$$

$$- \frac{1}{(n+1)\,!}\int_0^{\delta \log a} (\log w - \log \delta)^{n+1}\,e^{-w}\,dw$$

$$= \sum_{t=0}^{n+1} \frac{(-1)^t}{t\,!}\,\frac{(\log \delta)^t}{(n+1-t)\,!}\int_0^\infty e^{-w}\,(\log w)^{n+1-t}\,dw + R(\delta, a)$$

$$= \sum_{t=0}^{n+1} \frac{(-1)}{t\,!}\,\frac{\Gamma^{(n+1-t)}}{(n+1-t)\,!}\,(\log \delta)^t + R(\delta, a), \quad (67)$$

by virtue of (52). Now

$$|\,R(\delta, a)\,| \leqslant \frac{2^{n+1}}{(n+1)\,!}\sum_{l=0}^{n+1} |\log \delta\,|^{n+1-l}\int_0^{\delta \log a} |\log w\,|^l\,dw. \quad (68)$$

Also

$$\int_0^{\delta \log a} |\log w|^l \, dw \leqslant \Big[w \,|\log w|^l \Big]_0^{\delta \log a} + l \int_0^{\delta \log a} |\log w|^{l-1} \, dw$$

$$\leqslant \delta \log a \,|\log \delta|^l (1 + |\log \log a|)^l + l \int_0^{\delta \log a} |\log w|^{l-1} \, dw,$$

since $0 < \delta \leqslant 1/20$,

$$\leqslant \delta \,|\log \delta|^l \,\phi(n, a) + (n+1) \int_0^{\delta \log a} |\log w|^{l-1} \, dw$$

$$= \phi(n, a)\, \delta \,|\log \delta|^l,$$

after l steps. Hence from (68)

$$|R(\delta, a)| \leqslant \phi(n, a) \sum_{l=0}^{n+1} |\log \delta|^{n+1-l} . \,|\log \delta|^l . \,\delta$$

$$= \phi(n, a) \,|\log \delta|^{n+1} \,\delta. \tag{69}$$

The lemma follows immediately from (65)-(67) and (69).

Corollary.

$$\int_a^\infty \frac{(l_2 u)^n}{n\,!} \frac{1}{u \log u} \frac{du}{u^\delta} = \theta \left\{ \phi(n, a) \,|\log \delta|^{n+1} \right\}. \tag{70}$$

Lemma 11. *For every integer* $\mu \geqslant 1$,

$$J(\mu, \delta) = \int_{a_\mu}^\infty \frac{R_\mu(u)}{u^{2+\delta}} \, du$$

tends to a limit $l = l(u)$, *as* $\delta \to 0$, *where*

$$l(\mu) = \sum_{r=0}^{\mu} B_r \frac{(l_2 \, a_\mu)^{\mu - r}}{(\mu - r)\,!}, \tag{71}$$

where a_μ *is defined as in equation* (15).

Proof. We have

$$\sum_{q_\mu} \frac{1}{q_\mu^s} = s \int_{a_\mu}^\infty \frac{\pi_\mu(u)}{u^{1+s}} \, du$$

$$= \int_{a_\mu}^\infty \frac{\pi_\mu(u) \, du}{u^{2+\delta}} + \delta \int_{a_\mu}^\infty \frac{\pi_\mu(u) \, du}{u^{2+\delta}}$$

$$= \int_{a_\mu}^\infty \frac{\psi(\mu, u) \, du}{u^{2+\delta}} + \int_{a_\mu}^\infty \frac{R_\mu(u) \, du}{u^{2+\delta}} + \delta \int_{a_\mu}^\infty \frac{\pi_\mu(u) \, du}{u^{2+\delta}}$$

$$= \sum_{r=0}^{\mu-1} B_r \int_{a_\mu}^{\infty} \frac{(l_2 u)^{\mu-r-1}}{(\mu-r-1)\,!} \frac{1}{u \log u} \cdot \frac{du}{u^\delta} +$$

$$+ J(\mu, \delta) + \delta \int_{a_\mu}^{\infty} \frac{\pi_\mu(u)}{u^2+\delta} du$$

$$= \Sigma_1 + J(\mu, \delta) + \Sigma_2, \text{ say.} \tag{72}$$

Now for $x \geqslant 2$ and any μ we have by the Hardy-Ramanujan result

$$\pi_\mu(x) \leqslant c_{14} \frac{(l_2 x + c_{15})^{\mu-1}}{(\mu-1)\,!} \frac{x}{\log x}, \tag{73}$$

$\log \log x + c_{15}$ being positive. From (73) we get

$$\int_{a_\mu}^{\infty} \frac{\pi_\mu(u)\, du}{u^{2+\delta}} \leqslant \int_{2}^{\infty} \frac{\pi_\mu(u)\, du}{u^{2+\delta}} \leqslant c_{14} \int_{2}^{\infty} \frac{(l_2 u + c_{15})^{\mu-1}}{(\mu-1)\,!} \frac{1}{u \log u} \cdot \frac{du}{u^\delta}$$

$$\leqslant \phi(\mu) \sum_{r=0}^{\mu-1} \int_{2}^{\infty} \frac{(l_2 u)^{\mu-1-r}}{(\mu-1-r)\,!} \frac{1}{u \log u} \cdot \frac{du}{u^\delta}$$

as in (68), where $\phi(\mu)$ depends only on μ. Hence

$$\Sigma_2 \leqslant \delta\, \phi(\mu) \sum_{r=0}^{\mu-1} \int_{2}^{\infty} \frac{(l_2 u)^{\mu-1-r}}{(\mu-1-r)\,!} \frac{1}{u \log u} \cdot \frac{du}{u^\delta}$$

$$\leqslant \delta\, \phi(\mu) \sum_{r=0}^{\mu-1} |\log \delta|^{\mu-r}, \text{ by (70), since } a_\mu = \phi(\mu),$$

$$= \phi(\mu)\, |\log \delta|^{\mu}\, \delta, \tag{74}$$

$\phi(\mu)$ being independent of δ. Also, by Lemma 10,

$$\Sigma_1 = \sum_{r=0}^{\mu-1} B_r \left\{ \frac{-(l_2 a_\mu)^{\mu-r}}{(\mu-r)\,!} + \sum_{t=0}^{\mu-r} \frac{(-1)^t}{t\,!} \frac{\Gamma^{(\mu-r-t)}}{(\mu-r-t)\,!} (\log \delta)^t \right\} +$$

$$+ \theta\{\phi(\mu)\, |\log \delta|^{\mu+1} \delta\}$$

$$= - \sum_{r=0}^{\mu-1} B_r \frac{(l_2 a_\mu)^{\mu-r}}{(\mu-r)\,!} + \sum_{r=0}^{\mu-1} B_r \frac{\Gamma^{(\mu-r)}}{(\mu-r)\,!} +$$

$$+ \sum_{t=1}^{\mu} \frac{(-1)^t}{t\,!} (\log \delta)^t \sum_{r=0}^{\mu-t} B_r \frac{\Gamma^{(\mu-r-t)}}{(\mu-r-t)\,!} +$$

$$+ \theta\, \{\phi(\mu)\, |\log \delta|^{\mu+1} \delta\}, \tag{75}$$

since, by virtue of Lemma 2,

$$\sum_{r=0}^{\infty} |B_r| \quad \text{is convergent.} \tag{76}$$

Hence from (72), (74) and (75) we get

$$F_\mu(s) = \sum_{q_\mu} \frac{1}{q_\mu s}$$

$$= -\sum_{r=0}^{\mu-1} B_r \frac{(l_2 a_\mu)^{\mu-r}}{(\mu-r)!} + \sum_{r=0}^{\mu-1} B \frac{\Gamma^{(\mu-r)}}{(\mu-r)!} +$$

$$+ \sum_{t=1}^{\mu} \frac{(-1)^t}{t!} (\log \delta)^t \sum_{r=0}^{\mu-t} B_r \frac{\Gamma^{(\mu-r-t)}}{(\mu-r-t)!} +$$

$$+ J(\mu, \delta) + \theta \left\{\phi(\mu) |\log \delta|^{\mu+1} \delta\right\}. \quad (77)$$

But by Lemma (9) we have

$$F_\mu(s) = \sum_{r=0}^{\mu} B_r \frac{\Gamma^{(\mu-r)}}{(\mu-r)!} + \sum_{t=1}^{\mu} \frac{(-1)^t}{t!} (\log \delta)^t \sum_{r=0}^{\mu-t} B_r \frac{\Gamma^{(\mu-r-t)}}{(\mu-r-t)!} +$$

$$+ \theta \left\{\phi(\mu) |\log \delta|^{\mu-1} \delta\right\}. \quad (78)$$

Equating (77) and (78) and making $\delta \to 0$ we get the required result since $\phi(\mu)$ is independent of δ.

Lemma 12. *If* $1 \leqslant \mu \leqslant \nu'$ *and* (23) *holds, then*

$$\int_{a_\mu}^{\infty} \frac{R_\mu(u)\, du}{u^2} = \sum_{r=0}^{\mu} B_r \frac{(l_2\, a_\mu)^{\mu-r}}{(\mu-r)!}.$$

Proof. We have only to show, by virtue of Lemma 11, that if $1 \leqslant \mu \leqslant \nu'$ then

$$J(\mu, \delta) = \int_{a_\mu}^{\infty} \frac{R_\mu(u)\, du}{u^{2+\delta}}$$

is uniformly convergent for $\delta \geqslant 0$. If $\delta \geqslant 0$,

$$|J(\mu, \delta)| \leqslant \int_{a_\mu}^{\infty} \frac{|R_\mu(u)|}{u^2}\, du. \quad (79)$$

Also

$$\int_{a_\mu}^{\infty} \frac{|R_\mu(u)|\, du}{u^2} = \int_{a_\mu}^{e^{e^\mu}} + \int_{e^{e^\mu}}^{\infty},$$

and

$$\int_{e^{e^\mu}}^{\infty} \frac{|R_\mu(u)|\, du}{u^2} \leqslant c_{16}\, h(u) \int_{e^{e^\mu}}^{\infty} \frac{(l_2\, u)^\mu}{\mu!} \frac{du}{u\, (\log u)^2}, \quad (80)$$

by (23), since $\mu \leqslant \nu'$. The integral on the right side of (80) is convergent for every fixed μ, since

$$\frac{(l_2\mu)^\mu}{\mu\,!}\,\frac{1}{u(\log u)^2} = \frac{(l_2\,u)^{\mu+2}}{(\mu+2)!}\,\frac{1}{\mu(\log u)^2}\,\frac{(u+1)\,(u+2)}{(l_2u)^2} \leqslant \frac{(u+1)\,(u+2)}{u\log u\,(l_2u)^2}.$$

Hence (79) is convergent or $J(\mu, \delta)$ converges uniformly for $\delta \geqslant 0$.

LEMMA 13. (36)-(37) *are true.*

PROOF. We have for $x \geqslant 2$ and hence for $l_2x \geqslant \nu$,

$$\sum_{q_\nu \leqslant x} \frac{1}{q_\nu} = \frac{\pi_\nu(x)}{x} + \int_{a_\nu}^{x} \frac{\pi_\nu(u)\,du}{u^2}$$

$$= \int_{a_\nu}^{x} \frac{\psi(\nu, u)\,du}{u^2} + \int_{a_\nu}^{\infty} \frac{R_\nu(u)\,du}{u^2} - \int_{x}^{\infty} \frac{R_\nu(u)\,du}{u^2} + \frac{\pi_\nu(x)}{x}$$

$$= \sum_{r=0}^{\nu-1} B_r \int_{a_\nu}^{x} \frac{(l_2u)^{\nu-r-1}}{(\nu-r-1)\,!}\,\frac{du}{u\log u} + \int_{a_\nu}^{\infty} \frac{R_\nu(u)}{u^2}\,du -$$

$$- \int_{x}^{\infty} \frac{R_\nu(u)}{u^2}\,du + \frac{\pi_\nu(x)}{x}$$

$$= \sum_{r=0}^{\nu} B_r \frac{(l_2x)^{\nu-r}}{(\nu-r)\,!} - \int_{x}^{\infty} \frac{R_\nu(u)\,du}{u^2} + \frac{\pi_\nu(x)}{x}, \tag{81}$$

by Lemma 12. Also when $l_2x \geqslant \nu$, by virtue of (23)

$$\int_{x}^{\infty} \frac{|R_\nu(u)|\,du}{u^2} \leqslant c\,h(\nu) \int_{x}^{\infty} \frac{(l_2u)^\nu}{\nu\,!}\,\frac{du}{u(\log u)^2}$$

$$= c\,h(\nu)\left\{\frac{(l_2x)^\nu}{\nu\,!} + \frac{(l_2x)^{\nu-1}}{(\nu-1)\,!} + \dots + 1\right\}\cdot\frac{1}{\log x}$$

$$\leqslant c_{17}\,\frac{(l_2x)^\nu}{\nu\,!}\,\frac{\sqrt{\nu}\,h(\nu)}{\log x}, \tag{82}$$

by Lemma 3. Also by (73), since $\nu \leqslant l_2x$,

$$\frac{\pi_\nu(x)}{x} \leqslant c_{14}\,\frac{(l_2x + c_{15})^{\nu-1}}{(\nu-1)\,!}\,\frac{1}{\log x}$$

$$\leqslant c_{18}\,\frac{(l_2x)^{\nu-1}}{(\nu-1)\,!}\,\frac{1}{\log x}$$

$$\leqslant c_{18}\,\frac{(l_2x)^\nu}{\nu\,!}\,\frac{\sqrt{\nu}\,h(\nu)}{\log x}. \tag{83}$$

The lemma follows from (81)-(83) on taking $c_5 \geqslant c_{17} + c_{18}$.

ON A PROBLEM OF HARDY ON THE DISTRIBUTION OF INTEGERS HAVING A GIVEN NUMBER OF PRIME FACTORS II

By L. G. SATHE

[Received April 4, 1953]

CHAPTER 4

ON THE SUM $\sum_{q_\nu \leqslant x^a} \frac{(\log q_\nu)^n}{q_\nu}$

INTRODUCTION. As before ν' is the integer fixed by (23). The following additional notations will be used in this section :

$$S_\nu(x, n) = \sum_{q_\nu \leqslant x} (\log q_\nu)^n / q_\nu,$$

$$S(x, n) = S_1(x, n) = \sum_{p \leqslant x} (\log p)^n / p,$$

$$S_\nu(x) = S_\nu(x, 0);\ S(x) = S_1(x) = \sum_{p \leqslant x} 1/p,$$

$$g(\nu, x) = \frac{(l_2\, x)^{\nu+1}}{(\nu+1)\,!}\, \nu^{1/2}\, h(\nu).$$

For $n \geqslant 1$ we define

$$E_\nu(x, n) = S_\nu(x, n) - (\log x)^n \sum_{r=0}^{\nu-1} B_r \sum_{t=1}^{\nu-r} (-1)^{t-1} \frac{(l_2\, x)^{\nu-r-t}}{(\nu-r-t)\,!} \frac{1}{n^t}, \tag{84}$$

where b, b_1, ... are absolute positive constants.

The result that will be proved in this section is :

If (23) *holds*, $1 \leqslant \nu \leqslant \nu'$, $1 \leqslant n \leqslant 2\, l_2\, x$, *then when* $l_2\, x \geqslant \nu$,

$$|\, E_\nu(x, n)\,| \leqslant b (\log x)^{n-1}\, g(\nu, x), \tag{85}$$

where b *is any constant satisfying*

$$b \geqslant 2\,(c_5 + b_3) + 1; \tag{86}$$

c_5 *is the constant in equation* (37), *and* b_3 *is a constant satisfying*

$$S(x, n) = \frac{(\log x)^n}{n} + \theta\left\{b_3 (\log x)^{n-1}\right\},$$

for $x \geqslant 2$ *and* $n \geqslant 1$.

This is proved later. The proof of (85) will complete the investigation of (27).

THE METHOD. We have for $x \geqslant 2$,

$$S_\nu(x, n) = (\log x)^n S_\nu(x) - \int_{a_\nu}^{x} \frac{n(\log u)^{n-1}}{u} S_\nu(u)\, du$$

$$= (\log x)^n S_\nu(x) - \int_{a_\nu}^{x} \frac{n(\log u)^{n-1}}{u} \left\{\sum_{r=0}^{\nu} B_r \frac{(l_2 u)^{\nu-r}}{(\nu-r)!}\right\} du -$$

$$- \int_{a_\nu}^{x} \frac{n(\log u)^{n-1}}{u} E_\nu(u)\, du, \qquad \text{by (36).}$$

From this it is easily seen that to prove (85) it will be sufficient to prove :

$$\left| n\int_{a_\nu}^{x} \frac{(\log u)^{n-1}}{u} E_\nu(u)\, du \right| \leqslant b_1 (\log x)^{n-1} g(\nu, x), \tag{86'}$$

under the given conditions. It is, however, easy to see that (86′) cannot be proved by any obvious method from what we know of $E_\nu(x)$ by virtue of equation (37) which is the best possible result deducible from (23). The reason is, we do not know anything more about $|E_\nu(u)|$ in the interval $a_\nu \leqslant u < e^{e^\nu}$ than what can be deduced from the Hardy-Ramanujan result (73).

We proceed on quite different lines for the proof of (85). We show that :

If (23) *is true*, $1 \leqslant \nu \leqslant \nu'$, *then*

$$|E_\nu(x, n)| \leqslant b(\log x)^{n-1} g(\nu, x),$$

when $l_2 \geqslant \nu$, *for all values of* n (*integral*) *satisfying*

$$1 \leqslant n \leqslant 10(l_2 x - \nu \log 2). \tag{88}$$

(85) follows immediately from this, since $10(1-\log 2) > 2$,

(88) is proved by induction through ν using the transformation formula

$$\nu S_\nu(x, n) = \sum_{l=0}^{n} \binom{n}{l} \Big\{ \sum_{p \leqslant x^{1-a}} \frac{(\log p)^l}{p} S_{\nu-1}(x/p, n-l) +$$

$$+ \sum_{q_{\nu-1} \leqslant x^a} \frac{(\log q_{\nu-1})^{n-l}}{q_{\nu-1}} S(x/q_{\nu-1}, l) - S(x^{1-a}, l)\, S_{\nu-1}(x^a, n-l) \Big\} +$$

$$+ \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p^t \leqslant x} \frac{1}{p^t} S_{\nu-t}\left(\frac{x}{p^t}, n\right) + \theta\, \{b_2 (\log x)^{n-1} g(\nu-1, x)\}, \quad (87)$$

which is true for $1 \leqslant n \leqslant 10\, l_2 x$; $2 \leqslant \nu \leqslant l_2 x$. Hereafter empty sums when $\nu = 2$ will be considered to be zero.

Assuming that (88) is true for the index $(\nu - 1)$ we show that, for instance, if $1 \leqslant n \leqslant 10(l_2 x - \nu \log 2)$, $l_2 x \geqslant \nu \geqslant 2$, $u \leqslant x^{1-a}$, then

$$|E_{\nu-1}(x/u, n-l)| \leqslant b\, g(\nu-1, x) (\log \overline{x/u})^{n-l-1},$$

for $0 < l \leqslant n$.

This is just sufficient to use (87) for the proof of (88) with the index ν.

In this way we overcome most serious difficulties. Unfortunately we have to estimate the various summations in (87) by a very elaborate process. Instead of $\frac{1}{2}$ we may give to a any value satisfying $0 < -\log a < 1$. The dominant term on the right side of (87) must therefore be independent of a. This serves as an excellent check on the calculations.

LEMMA 14.

$$S(x, n) = \sum_{p \leqslant x} \frac{(\log p)^n}{p} = \frac{(\log x)^n}{n} + \theta\, \{b_3 (\log x)^{n-1}\}, \quad (90)$$

for $n \geqslant 1$ and $x \geqslant 2$.

PROOF. Since $S(x) = \sum_{p \leqslant x} 1/p$, we get

$$S(x, n) = \sum_{p \leqslant x} \frac{(\log p)^n}{p} = S(x) (\log x)^n - \int_2^x \frac{S(u)\, n\, (\log u)^{n-1}\, du}{u}$$

$$= S(x)(\log x)^n - \int_2^x (l_2 u + B)\, n \frac{(\log u)^{n-1}}{u}\, du + $$
$$+ \theta\left\{ b_4 \int_2^x \frac{n(\log u)^{n-2}}{u}\, du \right\},$$

since it is well known that

$$S(x) = l_2 x + B + \theta\,(b_4 \cdot 1/\log x) \quad \text{for } x \geqslant 2.$$

Hence

$$S(x, n) = S(x)(\log x)^n - (l_2 x + B)(\log x)^n +$$
$$+ (\log\log 2 + B)(\log 2)^n + \int_2^x \frac{(\log u)^{n-1}}{u}\, du +$$
$$+ \theta\left\{ b_4 \frac{n}{n-1} (\log x)^{n-1} \right\}, \qquad \text{if } n \geqslant 2,$$
$$= \theta\left\{ b_4 (\log x)^{n-1} \right\} + (\log\log 2 + B)(\log 2)^n +$$
$$+ \frac{(\log x)^n}{n} - \frac{(\log 2)^n}{n} + \theta\left\{ 2\, b_4 (\log x)^{n-1} \right\}, \quad \text{if } n \geqslant 2,$$
$$= \frac{(\log x)^n}{n} + \theta\left\{ (\log x)^{n-1} [3\, b_4 + \right.$$
$$\left. + (|\log\log 2| + |B|)\log 2 + \log 2] \right\}$$
$$= \frac{(\log x)^n}{n} + \theta\left\{ b_5 (\log x)^{n-1} \right\}, \qquad \text{if } n \geqslant 2.$$

Also if $n = 1$ it is well known that

$$S(x, 1) = \sum_{p \leqslant x} \frac{\log p}{p} = \log x + \theta\,(b_6).$$

Taking $b_3 \geqslant b_5 + b_6$, the lemma follows.

COROLLARY. (88) *is true when* $\nu = 1$. (91)

PROOF. $$b\, g(1, x) \geqslant \frac{b\,(l_2 x)^2}{2\,!} \geqslant b/2 \geqslant b_3,$$

when $l_2 x \geqslant 1$, by virtue of (86).

LEMMA 15. *If* $x \geqslant 2$, *then for all* $\nu \geqslant 1$ *and* $n \geqslant 0$,

$$S_\nu(x, n) \leqslant b_7 \frac{(l_2 x + b_8)^\nu}{\nu\,!} (\log x)^n\, \nu. \tag{92}$$

PROOF. By (73) we have for $x \geqslant 2$, and $\nu \geqslant 1$,

$$\pi_\nu(x) \leqslant b_9 \frac{(l_2 x + b_{10})^{\nu-1}}{(\nu-1)!} \frac{x}{\log x}, \tag{93}$$

where $b_9 = c_{14}$, $b_{10} = c_{15}$. Hence since $\log \log 2 + b_{10}$ is positive,

$$S_\nu(x) = \sum_{q_\nu \leqslant x} \frac{1}{q_\nu} = \frac{\pi_\nu(x)}{x} + \int_{a_\nu}^{x} \frac{\pi_\nu(u)}{u^2} du, \qquad \text{if } x \geqslant 2,$$

$$\leqslant b_9 \frac{(l_2 x + b_{10})^{\nu-1}}{(\nu-1)!} \frac{1}{\log x} + b_9 \int_2^x \frac{(l_2 u + b_{10})^{\nu-1}}{(\nu-1)!} \frac{du}{u \log u},$$

by (93),

$$\leqslant b_9 \frac{(l_2 x + b_{10})^{\nu-1}}{(\nu-1)!} \frac{1}{\log x} + b_9 \frac{(l_2 x + b_{10})^{\nu}}{\nu!},$$

since $\log \log 2 + b_{10} > 0$,

$$\leqslant b_9 \frac{(l_2 x + b_{10})^{\nu}}{\nu!} \left\{ 1 + \frac{\nu}{(\log \log 2 + b_{10})} \frac{1}{\log 2} \right\}$$

$$\leqslant b_{11} \frac{(l_2 x + b_{10})^{\nu}}{\nu!} \nu.$$

Hence

$$S_\nu(x, n) \leqslant (\log x)^n S_\nu(x) \leqslant b_{11} \frac{(l_2 x + b_{10})^{\nu}}{\nu!} (\log x)^n \nu.$$

Taking $b_7 \geqslant b_{11}$, $b_8 \geqslant b_{10}$ the result follows.

LEMMA 16. (Pillai) *Let P's and Q's be numbers which belong to particular classes ; $\chi(x)$ and $\psi(x)$ two functions of x, and*

$$F(x) = \sum_{P \leqslant x} \chi(P); \; G(x) = \sum_{Q \leqslant x} \psi(Q); \; 0 < \beta < 1.$$

Then

$$\sum_{P \leqslant x} \chi(P) G(x/P)$$
$$= \sum_{P \leqslant x^\beta} \chi(P)\, G(x/P) + \sum_{Q \leqslant x^{1-\beta}} \psi(Q)\, F(x/Q) - F(x^\beta)\, G(x^{1-\beta}). \tag{95}$$

I give the proof for the sake of completeness.

$$\sum_{P \leqslant x} \chi(P)\, G(x/P) = \sum_{P \leqslant x^\beta} + \sum_{x^\beta < P \leqslant x}. \tag{96}$$

Also

$$\sum_{x^\beta < P \leqslant x} \chi(P)\, G(x/P) = \sum_{x^\beta < P \leqslant x} \chi(P) \sum_{Q \leqslant x/P} \psi(Q)$$

$$= \sum_{Q \leqslant x^{1-\beta}} \psi(Q) \sum_{P \leqslant x/Q} \chi(P) - F(x^\beta)\, G(x^{1-\beta})$$

$$= \sum_{Q \leqslant x^{1-\beta}} \psi(Q)\, F(x/Q) - F(x^\beta)\, G(x^{1-\beta}). \qquad (97)$$

The second term on the right side of (97) corresponds to the lattice points (P, Q) in the rectangle $0 \leqslant P \leqslant x^\beta$; $0 \leqslant Q \leqslant x^{1-\beta}$. The result follows immediately from (96) and (97).

LEMMA 17. *If* $2 \leqslant \nu \leqslant l_2 x$ *and* $1 \leqslant n \leqslant 10\, l_2 x$, *then*

$$\nu\, S_\nu(x, n) = \sum_{l=0}^{n} \binom{n}{l} \sum_{p \leqslant x} \frac{(\log p)^l}{p} S_{\nu-1}(x/p, n-l) +$$

$$+ \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p^t \leqslant x} \frac{1}{p^t} S_{\nu-t}(x/p^t, n) +$$

$$+ \theta\left\{b_{12} (\log x)^{n-1} g(\nu-1, x)\right\}. \qquad (98)$$

PROOF. We have

$$\nu\, S_\nu(x, n) = \nu \sum_{q_\nu \leqslant x} \frac{(\log q_\nu)^n}{q_\nu}$$

$$= \sum_{t=1}^{\nu} (-1)^{t-1} \sum_{p^t \leqslant x} \sum_{q_{\nu-t} \leqslant x/p^t} \frac{(\log p^t + \log q_{\nu-t})^n}{p^t q_{\nu-t}}, \qquad (99)$$

with the convention $q_0 = 1$. For

$$\nu \sum_{q_\nu \leqslant x} \frac{(\log q_\nu)^n}{q_\nu} = \sum_{p \leqslant x} \sum_{\substack{q_{\nu-1} \leqslant x/p \\ (p, q_{\nu-1}) = 1}} \frac{(\log p + \log q_{\nu-1})^n}{p\, q_{\nu-1}}$$

$$= \sum_{p \leqslant x} \sum_{q_{\nu-1} \leqslant x/p} \frac{(\log p + \log q_{\nu-1})^n}{p\, q_{\nu-1}} - \sum_{p^2 \leqslant x} \sum_{\substack{q_{\nu-2} \leqslant x/p^2 \\ (p, q_{\nu-2}) = 1}} \frac{(\log p^2 + \log q_{\nu-2})^n}{p^2 q_{\nu-2}}.$$

Proceeding thus we get (99) as in Lemma 4. From (99) it follows that

$$\nu S_\nu(x, n) = \sum_{p \leqslant x} \sum_{q_{\nu-1} \leqslant x/p} \frac{(\log p + \log q_{\nu-1})^n}{p\, q_{\nu-1}} +$$

$$+ \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p^t \leqslant x} \frac{1}{p^t} S_{\nu-t}(x/p^t, n) +$$

$$+ \sum \cdots$$

$$= \sum_{l=0}^{n} \binom{n}{l} \sum_{p \leqslant x} \frac{(\log p)^l}{p} S_{\nu-1}(x/p, n-l)$$

$$+ \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p^t \leqslant x} \frac{1}{p^t} S_{\nu-t}(x/p^t, n) + \sum \cdots, \tag{100}$$

where

$$\sum = \sum_{t=2}^{\nu} (-1)^{t-1} \sum_{p^t \leqslant x} \sum_{q_{\nu-t} \leqslant x/p^t} \frac{(\log p^t + \log q_{\nu-t})^n - (\log q_{\nu-t})^n}{p^t\, q_{\nu-t}}$$

$$= \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p^t \leqslant x/2} \sum_{q_{\nu-t} \leqslant x/p} \frac{(\log p^t + \log q_{\nu-t})^n - (\log q_{\nu-t})^n}{p^t\, q_{\nu-t}} +$$

$$+ (-1)^{\nu-1} \sum_{p^\nu \leqslant x} \frac{(\log p^\nu)^n}{p^\nu}.$$

Now if $x \geqslant y > 0$, then $x^n - y^n = (x-y)(x^{n-1} + \ldots + y^{n-1})$

$$\leqslant n(x-y)\, x^{n-1}. \tag{100'}$$

Hence

$$\left|\sum\right| \leqslant n \sum_{t=2}^{\nu-1} \sum_{p^t \leqslant x/2} \frac{(\log p^t)}{p^t} \sum_{q_{\nu-t} \leqslant x/p^t} \frac{(\log p^t + \log q_{\nu-t})^{n-1}}{q_{\nu-t}} +$$

$$+ (\log x)^{n-1} \sum_{p^\nu \leqslant x} \frac{\log p^\nu}{p^\nu}$$

$$\leqslant n (\log x)^{n-1} \nu \sum_{t=2}^{\nu-1} \sum_{p^t \leqslant x/2} \frac{\log p}{p^t} \sum_{q_{\nu-t} \leqslant x/p^t} \frac{1}{q_{\nu-t}} +$$

$$+ (\log x)^{n-1} \nu \sum_{p^\nu \leqslant x} \frac{(\log p)}{p^\nu}$$

$$\leqslant n(\log x)^{n-1}\,\nu\sum_{t=2}^{\nu-1}\sum_{p^t\leqslant x/2}\frac{\log p}{p^t}\,.\,b_7\,\frac{(l_2\,x/p^t+b_8)^{\nu-t}\,\nu}{(\nu-t)\,!}+$$

$$+(\log x)^{n-1}\,\nu\sum_{p^\nu\leqslant x}\frac{\log p}{p^\nu},\qquad\text{by (92)},$$

$$<(b_7+1)\,n(\log x)^{n-1}\,\nu^2\,\frac{(l_2\,x+b_8)^{\nu-1}}{(\nu-1)\,!}\sum_{t=2}^{\nu}\sum_{p}\frac{\log p}{p^t},$$

as $\log\log 2+b_8>0$, and $\nu\leqslant l_2\,x$;

$$\leqslant b_{13}\,n(\log x)^{n-1}\,\nu^2\,\frac{(l_2\,x)^{\nu-1}}{(\nu-1)\,!}\sum_{t=2}^{\infty}\sum_{p}\frac{\log p}{p^t}$$

$$\leqslant b_{14}(\log x)^{n-1}\,\frac{(l_2\,x)^{\nu-1}}{(\nu-1)\,!}\,\nu^2\,l_2\,x,\qquad\text{as } n<10\,l_2\,x,$$

$$\leqslant b_{15}(\log x)^{n-1}\,\frac{(l_2\,x)^{\nu}}{\nu\,!}\,(\nu-1)^{1/2}\,h(\nu-1)\,,\tag{101}$$

as

$$h(\nu-1)\geqslant\prod_{m=1}^{\nu-1}\left(1+\frac{1}{\sqrt{m}}\right)\geqslant 16\,\nu^5,\ \text{for}\ \nu\geqslant 2.\tag{102}$$

Hence from (101) we get

$$\left|\sum\right|\leqslant b_{15}(\log x)^{n-1}\,g(\nu-1,x).$$

Taking $b_{12}=b_{15}$, the lemma follows.

LEMMA 18. (87) *holds.*

PROOF. In Lemma 16 put p's $=$ the prime numbers p, q's $=$ the set of integers $q_{\nu-1}$, and

$$\chi(x)=\frac{(\log x)^l}{x},\ \psi(x)=\frac{(\log x)^{n-l}}{x};\ \beta=1-\alpha.$$

The result then follows immediately from (98) by taking $b_2\geqslant b_{12}$.

LEMMA 19. *If*

$$\left.\begin{array}{l}2\leqslant\nu\leqslant\nu^1,\ 1\leqslant n\leqslant 10(l_2\,x-\nu\log 2)\\ \textit{and (88) is true for the index }(\nu-1),\end{array}\right\}\tag{103}$$

then, when $l_2 x > \nu$,

$$\sum_{l=0}^{n} \binom{n}{l} \sum_{p \leqslant x^{1-a}} \frac{(\log p)^l}{p} S_{\nu-1}(x/p, n-l)$$

$$= \sum_{l=0}^{n-1} \binom{n}{l} \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \times$$

$$\times \sum_{p \leqslant x^{1-a}} \frac{(\log p)^l}{p} \frac{(\log x - \log p)^{n-l}}{(n-l)\,t} \frac{(l_2 \overline{x/p})^{\nu-r-t-1}}{(\nu-r-t-1)!} +$$

$$+ \sum_{r=0}^{\nu-1} B_r \sum_{p \leqslant x^{1-a}} \frac{(\log p)^n}{p} \frac{(l_2 \overline{x/p})^{\nu-r-1}}{(\nu-r-1)!} +$$

$$+ \theta \left\{ b (\log v)^{n-1} g(\nu-1, x)\, l_2 x \left(1 + \frac{b_{17}}{\nu}\right) \right\}. \quad (104)$$

PROOF. By virtue of equations (36) and (84) it is easily seen on changing the order of summation with respect to p and l, r, t that we have only to show that

$$\left| \sum_{l=0}^{n} \binom{n}{l} \sum_{p \leqslant x^{1-a}} \frac{(\log p)^l}{p} E_{\nu-1}(x/p, n-l) \right|$$

$$< b(\log x)^{n-1} g(\nu-1, x)\, l_2 x \left(1 + \frac{b_{17}}{\nu}\right). \quad (105)$$

We have, by virtue of (103) and (88), if $u \leqslant x^{1-a}$,

$$|E_{\nu-1}(x/u, n-l)| \leqslant b(\log \overline{x/u})^{n-l-1} g(\nu-1, x/u)$$

$$\leqslant b(\log \overline{x/u})^{n-l-1} g(\nu-1, x),$$

for $0 \leqslant l \leqslant n-1$, if $1 \leqslant n-l \leqslant 10\left\{l_2 x^a - \overline{\nu-1} \log 2\right\}$, $l_2 x^a > \nu - 1$; *a fortiori*, when $1 \leqslant n \leqslant 10\,(l_2 x - \nu \log 2)$, and $l_2 x > \nu$, $0 \leqslant l \leqslant n-1$. Also since $2 \leqslant \nu \leqslant \nu'$, we have, by virtue of (37),

$$|E_{\nu-1}(x/u, 0)| < c_5 \frac{(l_2 \overline{x/u})^{\nu-1}}{(\nu-1)!} (\nu-1)^{1/2} \frac{h(\nu-1)}{(\log \overline{x/u})},$$

if $l_2 \overline{x/u} > \nu - 1$. Hence if $l_2 x > \nu$, and $u \leqslant x^{1-a}$,

$$|E_{\nu-1}(x/u, 0)| < c_5 \frac{(l_2 x)^\nu}{\nu!} (\nu-1)^{1/2} h(\nu-1) (\log \overline{x/u})^{-1}$$

$$\leqslant b\, g(\nu-1, x) (\log \overline{x/u})^{-1}, \quad \text{by (86).}$$

Hence, under the conditions (103),

$$|E_{\nu-1}(x/u, n-l)| \leqslant b.\, g(\nu-1, x)(\log x/u)^{n-l-1}, \quad (106)$$

for $0 \leqslant l \leqslant n$ and $1 \leqslant u \leqslant x^{1-a}$.

$$\left|\sum_{l=0}^{n}\binom{n}{l}\sum_{p\leqslant x^{1-a}}\frac{(\log p)^l}{p}E_{\nu-1}(x/p, n-l),\right|$$

$$\leqslant b\, g(\nu-1, x)\sum_{l=0}^{n}\binom{n}{l}\sum_{p\leqslant x^{1-a}}\frac{(\log p)^l}{p}\frac{(\log \overline{x/p})^{n-l}}{(\log x/p)}, \text{ by (106).}$$

$$= b\, g(\nu-1, x)\sum_{p\leqslant x^{1-a}}\frac{(\log x)^n}{p(\log x/p)}, \quad (108)$$

by the bionomial theorem.

Also

$$\sum_{p\leqslant x^{1-a}}\frac{1}{p\log x/p} \leqslant \sum_{p\leqslant x^{1-a}}\left\{\frac{1}{p\log x}+\frac{2\log p}{p(\log x)^2}\right\}$$

$$\leqslant \frac{l_2 x + b_{18}}{\log x}(\ldots) \leqslant \frac{l_2 x}{\log x}\left(1+\frac{b_{18}}{\nu}\right).$$

From this and (108) the result follows on taking $b_{17} > b_{18}$.

LEMMA 20. *If*

$$T(\nu-1, l) = T(\nu-1, l, x^{1-a})$$

$$= \sum_{p\leqslant x^{1-a}}\frac{(\log p)^l(\log x - \log p)^{n-l}}{p}\frac{(l_2\overline{x/p})^{\nu-1}}{(\nu-1)!}, \quad (109)$$

then for $l_2 x \geqslant \nu \geqslant 1$, $1 \leqslant n \leqslant 10\, l_2 x$, and $t \geqslant 1$,

$$\sum_{l=1}^{n-1}\binom{n}{l}\frac{1}{(n-l)^t}T(\nu-1, l) = (\log x)^n\sum_{l=1}^{n-1}\binom{n}{l}\frac{1}{(n-l)^t}I_{\nu-1,\, l}+$$

$$+\theta\left\{b_{19}\frac{(l_2 x)^{\nu+1}}{(\nu+1)!}(\log x)^{n-1}\nu^2\right\}, \quad (110)$$

where

$$I_{\nu-1, l} = \sum_{s=1}^{l}(-1)^{s-1}\binom{l-1}{s-1}\times$$

$$\times\sum_{t'=1}^{\nu}(-1)^{t'-1}\frac{(l_2 x)^{\nu-t'}}{(\nu-t')!}\frac{1}{(n-l+s)^{t'}}+$$

$$+\sum_{s=1}^{l}(-1)^s\binom{l-1}{s-1}\sum_{t'=1}^{\nu}(-1)^{t'-1}\frac{(l_2-x^a)^{\nu-t'}}{(\nu-t')!}\cdot\frac{a^{n-l+s}}{(n-l+s)^{t'}}. \quad (110')$$

PROOF. Since

$$\frac{d}{du}\left\{\frac{(\log u)^{l-1}}{u}\right\}=(l-1)\frac{(\log u)^{l-2}}{u^2}-\frac{(\log u)^{l-1}}{u^2},$$

$\dfrac{(\log u)^{l-1}}{u}$ decreases if $\log u \geqslant -1$. Hence

$$F(u)=F(u,x)=\frac{(\log u)^{l-1}}{u}(\log x-\log u)^{n-l}\frac{(l_2\,\overline{x/u})^{\nu-1}}{(\nu-1)!} \quad (111)$$

is a decreasing function of u for $e^l \leqslant u \leqslant \sqrt{x}$ if

$$1 \leqslant l \leqslant n \leqslant 10\,l_2\,x; 1 \leqslant \nu \leqslant l_2\,x;\ l_2\,x \geqslant 5.$$

It may be mentioned explicitly that

$$e^{10l_2x} \leqslant \sqrt{x},\ \text{when}\ l_2\,x \geqslant 5. \quad (111')$$

Now by (111′) we have, if $1 \leqslant l \leqslant n \leqslant 10\,l_2\,x$, $1 \leqslant \nu \leqslant l_2\,x$, $l_2\,x \geqslant 5$, then

$$T(\nu-1,l)=\sum_{p\leqslant x^{1-a}}\frac{(\log p)^l}{p}(\log x-\log p)^{n-l}\frac{(l_2\,\overline{x/p})^{\nu-1}}{(\nu-1)!}$$

$$=\sum_{p<e^l}+\sum_{e^l\leqslant p\leqslant x^{1-a}}=\Sigma_{1,l}+\Sigma_{2,l}\text{, say.} \quad (112)$$

$$|\Sigma_{1,l}| \leqslant (\log x)^{n-l}\frac{(l_2\,x)^{\nu-1}}{(\nu-1)!}\sum_{p\leqslant e^l}\frac{(\log p)^l}{p}$$

$$\leqslant b_{21}(\log x)^{n-l}\frac{(l_2\,x)^{\nu-1}}{(\nu-1)!}\,l^l, \quad (113)$$

since, by Lemma 14,

$$\sum_{p\leqslant x}\frac{(\log p)^n}{p} \leqslant b_{21}(\log x)^n,\ \text{for}\ n \geqslant 1\ \text{and}\ x \geqslant 2. \quad (113')$$

Hence by (113), if $t \geqslant 1$,

$$\sum_{l=1}^{n-1}\binom{n}{l}\frac{1}{(n-l)^t}|\Sigma_{1,l}| \leqslant b_{21}\frac{(l_2\,x)^{\nu-1}}{(\nu-1)!}\sum_{l=1}^{n-1}\binom{n}{l}(\log x)^{n-l}\,l^l. \quad (114)$$

Also

$$\sum_{l=1}^{n-1}\binom{n}{l}(\log x)^{n-l}\,l^l < \sum_{l=1}^{n}\binom{n}{l}(\log x)^{n-l}(10\,l_2\,x)^l$$

$$= (\log x + 10\,l_2\,x)^n - (\log x)^n$$

$$\leqslant 10\,l_2\,x\,(\log x + 10\,l_2\,x)^{n-1}\cdot n, \qquad \text{by } (100');$$

$$\leqslant 100\,(l_2\,x)^2(\log x)^{n-1}\left(1+\frac{10\,l_2\,x}{\log x}\right)^{10\,l_2 x}$$

$$\leqslant b_{22}\,(l_2\,x)^2(\log x)^{n-1}. \tag{115}$$

From (114) and (115),

$$\sum_{l=1}^{n-1}\binom{n}{l}\frac{1}{(n-l)^t}\sum\nolimits_{1,l} < b_{23}\,\frac{(l_2\,x)^{\nu+1}}{(\nu+1)\,!}(\log x)^{n-1}\,\nu^2. \tag{116}$$

We write now, as usual,

$$g(x) = \sum_{p\leqslant x}\log p = x + \epsilon(x);$$

then for $x > 2$ it is well known that

$$|\,\epsilon(x)\,| \leqslant b_{24}\,\frac{x}{(\log x)^2}. \tag{117}$$

Now

$$\sum\nolimits_{2,l}$$

$$= \sum_{e^l\leqslant p\leqslant x^{1-a}}\frac{(\log p)^l}{p}(\log x - \log p)^{n-l}\,\frac{(l_2\,\overline{x/p})^{\nu-1}}{(\nu-1)\,!}$$

$$= \sum_{e^l\leqslant m\leqslant x^{1-a}}\frac{(\log m)^l}{m}(\log x - \log m)^{n-l}\frac{(l_2\,\overline{x/m})^{\nu-1}}{(\nu-1)\,!}\left(\frac{g(m)-g(m-1)}{\log m}\right)$$

$$= \sum_{e^l\leqslant m\leqslant x^{1-a}} F(m) + \sum_{e^l\leqslant m\leqslant x^{1-a}}\{\epsilon(m)-\epsilon(m-1)\}\,F(m), \tag{118}$$

where $F(u) = F(u,x)$ is defined in (111). Now,

$$\Big|\sum_{e^l\leqslant m\leqslant x^{1-a}}\{\epsilon(m)-\epsilon(m-1)\}\,F(m)\,\Big|$$

$$< \Big|\sum_{e^l\leqslant m\leqslant x^{1-a}}\epsilon(m)\{F(m)-F(m+1)\}\Big| +$$

$$+\, b_{25}\,F(e^l)\,e^l + b_{25}\,F(x^{1-a})\,\frac{x^{1-a}}{(\log x)^2}, \qquad \text{by } (117),$$

$$< b_{26} \sum_{e^l \leqslant m \leqslant x^{1-a}} \frac{m}{(\log m)^2} \{F(m) - F(m+1)\} +$$

$$+ b_{26} F(e^l) e^l + b_{26} F(x^{1-a}) \frac{x^{1-a}}{(\log x)^2}, \qquad \text{by (111) and (117),}$$

$$< b_{27} \sum_{e^l \leqslant m \leqslant x^{1-a}} F(m) \left\{ \frac{m}{(\log m)^2} - \frac{\overline{m-1}}{(\log \overline{m-1})^2} \right\} +$$

$$+ b_{27} l^l (\log x)^{n-l} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} + b_{27} (\log x)^{n-1} (1-a)^l a^{n-l} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!}$$

$$< b_{28} \sum_{2 \leqslant m \leqslant x^{1-a}} \frac{F(m)}{(\log m)^2} +$$

$$+ b_{28} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \{ l^l (\log x)^{n-l} + (\log x)^{n-1} (1-a)^l a^{n-l} \}$$

$$< b_{28} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \sum_{2 \leqslant m \leqslant x^{1-a}} \frac{(\log m)^{l-1} (\log \overline{x/m})^{n-l}}{m (\log m)^2} +$$

$$+ b_{28} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \{ l^l (\log x)^{n-l} + (\log x)^{n-1} (1-a)^l a^{n-l} \}$$

$$= b_{28} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \sum_{2 \leqslant m \leqslant x^{1-a}} \frac{(\log m)^l (\log \overline{x/m})^{n-l}}{m (\log m)^3} +$$

$$+ b_{28} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \{ l^l (\log x)^{n-l} + (\log x)^{n-1} (1-a)^l a^{n-l} \}. \qquad (119)$$

Hence from (119) we get

$$\left| \sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-1)^l} \sum_{e^l \leqslant m \leqslant x^{1-a}} \{ \epsilon(m) - \epsilon(m-1) \} F(m) \right|$$

$$< b_{28} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \sum_{2 \leqslant m \leqslant x^{1-a}} \sum_{l=1}^{n} \frac{\binom{n}{l} (\log m)^l (\log \overline{x/m})^{n-l}}{m (\log m)^3} +$$

$$+ b_{28} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \sum_{l=1}^{n-1} \binom{n}{l} l^l (\log x)^{n-l} +$$

$$+ b_{28} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^{n-1} \sum_{l=0}^{n} \binom{n}{l} (1-a)^l a^{n-l}$$

$$\leqslant b_{29} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \sum_{2 \leqslant m \leqslant x^{1-a}} \frac{(\log x)^n - (\log \overline{x/m})^n}{m (\log m)^3} +$$

$$+ b_{29} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (l_2 x)^2 (\log x)^{n-1} + b_{29} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^{n-1},$$

by (115), since $\sum_{l=0}^{n} \binom{n}{l} (1-a)^l a^{n-l} = 1,$

$$\leqslant b_{30} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (l_2 x)^2 (\log x)^{n-1}, \qquad \text{by } (100'),$$

$$\leqslant b_{31} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} (\log x)^{n-1} \nu^2, \tag{120}$$

since $\nu \leqslant l_2 x$; $n \leqslant 10\, l_2 x$.

From (118) and (120) we get

$$\sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^l} \Sigma_{2,l} = \sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^l} \sum_{e^l \leqslant m \leqslant x^{1-a}} F(m, x) +$$
$$+ \theta \left\{ b_{31} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} (\log x)^{n-1} \nu^2 \right\}. \tag{121}$$

Now,

$$\sum_{e^l \leqslant m \leqslant x^{1-a}} F(m, x) = \int_{e^l}^{x^{1-a}} F(u, x)\, du + \theta \left\{ b_{32}\, l^l (\log x)^{n-l} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \right\},$$

by (111),

$$= \int_{x^a}^{x/e^l} \frac{(l_2 v)^{\nu-1}}{(\nu-1)!} (\log v)^{n-l} \frac{(\log x - \log v)^{l-1}}{v}\, dv +$$
$$+ \theta \left\{ b_{32}\, l^l (\log x)^{n-l} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \right\}$$

$$= \int_{x^a}^{x} - \int_{x/e^l}^{x} + \theta \left\{ b_{32}\, l^l (\log x)^{n-l} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \right\}. \tag{122}$$

$$\left| \int_{x/e^l}^{x} \frac{(l_2 v)^{\nu-1}}{(\nu-1)!} (\log v)^{n-l} \frac{(\log x - \log v)^{l-1}}{v}\, dv \right|$$
$$\leqslant \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^{n-l} \left[- \frac{(\log x - \log v)^l}{l} \right]_{x/e^l}^{x} \leqslant \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^{n-l} l^l. \tag{123}$$

Hence from (121)-(123) and (115) we get

$$\sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^l} \sum_{2,l}$$

$$= \sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^l} \int_{x^a}^{x} \frac{(l_2 v)^{\nu-1}}{(\nu-1)!} (\log v)^{n-l} \frac{(\log x - \log v)^{l-1}}{v} dv +$$

$$+ \theta \left\{ b_{33} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} (\log x)^{n-1} \nu^2 \right\}. \tag{124}$$

Hence from (112), (116) and (124), if $l_2 x > 5$; $1 \leqslant \nu \leqslant l_2 x$; $1 \leqslant n \leqslant 10\, l_2 x$,

$$\sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^l} T(\nu-1, l) = \sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^l} \left\{ \sum_{1,l} + \sum_{2,l} \right\}$$

$$= \sum_{l=1}^{n-1} \binom{n}{l} \frac{(\log x)^n}{(n-l)^l} \int_a^1 \frac{(\log \overline{z \log x})^{\nu-1}}{(\nu-1)!} (1-z)^{l-1} z^{n-l} dz +$$

$$+ \theta \left\{ b_{34} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} (\log x)^{n-1} \nu^2 \right\}, \tag{124'}$$

on transforming the integral in (124) by substituting $\log v = z \log x$.

Also for $1 \leqslant l_2 x < 5$; $1 \leqslant \nu < l_2 x$; $1 \leqslant n \leqslant 10\, l_2 x$,

$$\sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^l} T(\nu-1, l) - \sum_{l=1}^{n-1} \binom{n}{l} \frac{(\log x)^n}{(n-l)^l} \times$$

$$\times \int_a^1 \frac{(\log \overline{z \log x})^{\nu-1}}{(\nu-1)!} (1-z)^{l-1} z^{n-l} dz$$

$$= \theta\,(b_{35}) = \theta \left\{ b_{36} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} (\log x)^{n-1} \nu^2 \right\}. \tag{124''}$$

Hence from (124′) and (124″) if $l_2 x > \nu > 1$, $1 < n \leqslant 10\, l_2 x$, $t > 1$

$$\sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^l} T(\nu-1, l)$$

$$= \sum_{l=1}^{n-1} \binom{n}{l} \frac{(\log x)^n}{(n-l)^l} \int_a^1 \frac{(\log \overline{z \log x})^{\nu-1}}{(\nu-1)!} (1-z)^{l-1} z^{n-l} dz +$$

$$+ \theta \left\{ (b_{34} + b_{36}) \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} (\log x)^{n-1} \nu^2 \right\}, \tag{125}$$

Now for $\nu > 1$

$$\int \frac{(\log \overline{z \log x})^{\nu-1}}{(\nu-1)\,!}\, z^m\, dz = \sum_{t'=1}^{\nu} (-1)^{t'-1} \frac{(\log \overline{z \log x})^{\nu-t'}}{(\nu-t')\,!} \frac{z^{m+1}}{(m+1)^{t'}}. \tag{125'}$$

Hence

$$\int \frac{(\log \overline{z \log x})^{\nu-1}}{(\nu-1)\,!} (1-z)^{l-1} z^{n-l}\, dz$$

$$= \sum_{s=1}^{l} (-1)^{s-1} \binom{l-1}{s-1} \int \frac{(\log \overline{z \log x})^{\nu-1}}{(\nu-1)\,!} z^{n-l+s-1}\, dz$$

$$= \sum_{s=1}^{l} (-1)^{s-1} \binom{l-1}{s-1} \sum_{t'=1}^{\nu} (-1)^{t'-1} \frac{(\log \overline{z \log x})^{\nu-t'}}{(\nu-t')\,!} \frac{z^{n-l+s}}{(n-l+s)^{t'}}. \tag{126}$$

The lemma follows from (125) and (126) taking $b_{19} > b_{34} + b_{36}$.

LEMMA 21. *If* $2 \leqslant \nu \leqslant l_2 x$, $1 \leqslant n \leqslant 10\, l_2 x$, *then*

$$\sum_{l=1}^{n-1} \binom{n}{l} \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} \frac{(-1)^{t-1}}{(n-l)^t} T(\nu-r-t-1, l)$$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)\,!} \times$$

$$\times \sum_{l=1}^{n-1} \sum_{s=1}^{l} \frac{(-1)^{s-1}}{s} \binom{l-1}{s-1} \binom{n}{l} \left\{ \frac{1}{(n-l)^t} - \frac{1}{(n-l+s)^t} \right\} +$$

$$+ (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)\,!} \times \cdot$$

$$\times \sum_{l=1}^{n-1} \sum_{s=1}^{l} \frac{(-1)^s}{s} \binom{l-1}{s-1} \binom{n}{l} a^{n-l+s} \left\{ \frac{1}{(n-l)^t} - \frac{1}{(n-l+s)^t} \right\} +$$

$$+ \theta \left\{ b_{37} \frac{(l_2 x)^{\nu+1}}{(\nu+1)\,!} (\log x)^{n-1} \nu^3 \right\}. \tag{127}$$

PROOF. Changing the order of summation the given sum is equal to

$$\sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^t} T(\nu - r - t - 1, l)$$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^t} I_{\nu-r-t-1,\,l} +$$

$$+ \theta \left\{ b_{19} (\log x)^{n-1} \nu^2 \sum_{r=0}^{\nu-2} |B_r| \sum_{t=1}^{\nu-r-1} \frac{(l_2 x)^{\nu-r-t+1}}{(\nu-r-t+1)\,!} \right\}, \qquad (128)$$

by Lemma 20, since $l_2 x \geqslant \nu - r - t - 1 + 1 \geqslant 1$ for the values of r and t in the above summation. Now $l_2 x \geqslant \nu$ and $\sum\limits_{r=0}^{\infty} |B_r|$ is convergent by Lemma 2. Hence

$$\sum_{r=0}^{\nu-2} |B_r| \sum_{t=1}^{\nu-r-1} \frac{(l_2 x)^{\nu-r-t+1}}{(\nu-r-t+1)\,!} \leqslant 2 \frac{(l_2 x)^{\nu+1}}{(\nu+1)\,!} \nu \sum_{r=0}^{\infty} |B_r| .$$

Hence the given sum is equal to

$$(\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^t} I_{\nu-r-t-1,\,l} +$$

$$+ \theta \left\{ b_{37} \frac{(l_2 x)^{\nu+1}}{(\nu+1)\,!} (\log x)^{n-1} \nu^3 \right\}. \qquad (129)$$

Now

$$\sum_{t=1}^{\nu-r-1} (-1)^{t-1} \sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^t} I_{\nu-r-t-1,\,l}$$

$$= \sum_{l=1}^{n-1} \binom{n}{l} \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{1}{(n-l)^t} I_{\nu-r-t-1,\,l} . \qquad (130)$$

Contribution to (130) corresponding to the second line in the equation (110′) giving the value of $I_{\nu-r-t-1,\,l}$ is equal to

$$\sum_{l=1}^{n-1} \binom{n}{l} \sum_{s=1}^{l} (-1)^s \binom{l-1}{s-1} \sum_{t=0}^{\nu-r-1} \sum_{t'=1}^{\nu-r-t} \frac{(-1)^{t-1} (-1)^{t'-1}}{(n-l)^t} \times$$

$$\times \frac{(l_2 x^a)^{\nu-r-t-t'}}{(\nu-r-t-t')} \frac{a^{n-l+s}}{(n-l+s)^{t'}}$$

$$= \sum_{l=1}^{n-1} \binom{n}{l} \sum_{s=1} (-1)^s \binom{l-1}{s-1} a^{n-l+s} \sum_{\lambda=1}^{\nu-r-1} (-1)^{\lambda-1} \frac{(l_2 x^a)^{\nu-r-\lambda-1}}{(\nu-r-\lambda-1)!} \times$$

$$\times \sum_{t=1}^{\lambda} \frac{1}{(n-l)^t (n-l+s)^{\lambda-t+1}}$$

$$= \sum_{l=1}^{n-1} \binom{n}{l} \sum_{s=1}^{l} (-1)^s \binom{l-1}{s-1} \frac{a^{n-l+s}}{s} \sum_{\lambda=1}^{\nu-r-1} (-1)^{\lambda-1} \frac{(l_2 x^a)^{\nu-r-\lambda-1}}{(\nu-r-\lambda-1)!} \times$$

$$\times \left\{ \frac{1}{(n-l)^\lambda} - \frac{1}{(n-l+s)^\lambda} \right\}, \qquad (131)$$

since

$$\sum_{t=1}^{\lambda} \frac{1}{(n-l)^t (n-l+s)^{\lambda-t+1}} = \frac{1}{(n-l+s)^{\lambda+1}} \sum_{t=1}^{\lambda} \left(\frac{n-l+s}{n-l} \right)^t$$

$$= \frac{1}{(n-l+s)^{\lambda+1}} \, \frac{n-l+s}{s} \left\{ \frac{(n-l+s)^\lambda}{(n-l)^\lambda} - 1 \right\}.$$

The right side of (131) is equal to

$$\sum_{\lambda=1}^{\nu-r-1} (-1)^{\lambda-1} \frac{(l_2 x^a)^{\nu-r-\lambda-1}}{(\nu-r-\lambda-1)!} \sum_{l=1}^{n-1} \sum_{s=1}^{l} \frac{(-1)^s}{s} \binom{l-1}{s-1} \binom{n}{l} a^{n-l+s} \times$$

$$\times \left\{ \frac{1}{(n-l)^\lambda} - \frac{1}{(n-l+s)^\lambda} \right\}$$

$$= \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{l=1}^{n-1} \sum_{s=1}^{l} \frac{(-1)^s}{s} \binom{l-1}{s-1} \binom{n}{l} a^{n-l+s} \times$$

$$\times \left\{ \frac{1}{(n-l)^t} - \frac{1}{(n-l+s)^t} \right\}. \qquad (132)$$

Hence from (130)-(132) the contribution to

$$(\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \sum_{l=1}^{n-1} \binom{n}{l} \frac{1}{(n-l)^t} I_{\nu-r-t-1,l}$$

corresponding to the second line in the equation (110′) giving the value of $I_{\nu-r-t-1,l}$ is equal to the second line of the lemma i. e. equation (127). The first line in (127) is obtained by putting $a = 1$ in the second line and changing the sign, by the very nature of (110′). Hence the lemma follows from this and from (129).

Lemma 22. *If* $2 \leqslant \nu \leqslant l_2 x$, $1 \leqslant n \leqslant 10\, l_2 x$ *then*

$$\sum_{r=0}^{\nu-1} B_r T(\nu - r - 1, n)$$

$$= (\log x)^n \sum_{r=0}^{\nu-1} B_r \sum_{t=0}^{\nu-r-1} (-1)^t \frac{(l_2 x)^{\nu-r-t-1}}{(\nu - r - t - 1)!} \times$$

$$\times \sum_{s=0}^{n} \frac{(-1)^{s-1}}{s} \binom{n-1}{s-1} \cdot \frac{1}{s^t} +$$

$$+ (\log x)^n \sum_{r=0}^{\nu-1} B_r \sum_{t=0}^{\nu-r-1} (-1)^t \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{s=1}^{n} \frac{(-1)^s}{s} \binom{n-1}{s-1} \cdot \frac{a^s}{s^t} +$$

$$+ \theta \left\{ b_{38} \frac{(l_2 x)^{\nu+1}}{(\nu + 1)!} (\log x)^{n-1} \nu^2 \right\}. \qquad (133)$$

Proof. Using the notations of (112) we have for $1 \leqslant \nu \leqslant l_2 x$; $1 \leqslant n \leqslant 10 l_2 x$; $l_2 x \geqslant 5$,

$$T(\nu - 1, n) = \sum\nolimits_{1,n} + \sum\nolimits_{2,n}. \qquad (134)$$

Now by (113) which is true when $l = n$ also

$$\left|\sum\nolimits_{1,n}\right| \leqslant b_{21} \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} n^n \leqslant b_{39} (\log x)^{n-1} \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} l_2 x$$

since $n \leqslant 10\, l_2 x < \log x$ if $l_2 x \geqslant 5$,

$$\left|\sum\nolimits_{1,n}\right| \leqslant b_{40} (\log x)^{n-1} \cdot \frac{(l_2 x)^{\nu+1}}{(\nu + 1)!} \nu^2. \qquad (135)$$

Also by (118) and (119) we get

$$\sum\nolimits_{2,n} = \sum_{e^n \leqslant m \leqslant x^{1-a}} \frac{(\log m)^{n-1}}{m} \cdot \frac{(l_2\, \overline{x/m})^{\nu-1}}{(\nu - 1)!} +$$

$$+ \theta \left\{ \frac{(l_2 x)^{\nu-1}}{(\nu - 1)!} b_{28} \left(n^n + (\log x)^{n-1} + (\log x)^{n-1} \sum_{m=2}^{\infty} \frac{1}{m (\log m)^2} \right) \right.$$

$$= \int_{x^a}^{x} \frac{(l_2 v)^{\nu-1}}{(\nu - 1)!} \frac{(\log x - \log v)^{n-1}}{v} dv +$$

$$+ \theta \left\{ b_{41} (\log x)^{n-1} (l_2 x)^{\nu+1} \nu^2 / (\nu + 1)! \right\},$$

as in (135) on using equations (122), (123). Hence by (134) for

$l_2 x \geqslant \nu \geqslant 1$, $l_2 x \geqslant 5$, $1 \leqslant n \leqslant 10\, l_2 x$ we get as in (124′)

$$T(\nu - 1, n) = (\log x)^n \int_a^1 \frac{(\log \overline{z \log x})^{\nu-1}}{(\nu - 1)\,!} (1 - z)^{n-1} dz +$$
$$+ \theta \left\{ b_{42} (\log x)^{n-1} \frac{(l_2 x)^{\nu+1}}{(\nu + 1)\,!} \nu^2 \right\}.$$

Also for $l_2 x \geqslant \nu \geqslant 1$; $1 \leqslant l_2 x \leqslant 5$, $1 \leqslant n \leqslant 10\, l_2 x$, we get as in (124″)

$$T(\nu - 1, n) = (\log x)^n \int_a^1 \frac{(\log \overline{z \log x})^{\nu-1}}{(\nu - 1)\,!} (1 - z)^{n-1} dz +$$
$$+ \theta \left\{ b_{43} (\log x)^{n-1} \frac{(l_2 x)^{\nu+1}}{(\nu + 1)\,!} \nu^2 \right\}.$$

Hence for $l_2 x \geqslant \nu \geqslant 1$; $1 \leqslant n \leqslant 10\, l_2 x$,

$$T(\nu - 1, n) = (\log x)^n \int_a^1 \frac{(\log \overline{z \log x})^{\nu-1}}{(\nu - 1)\,!} (1 - z)^{n-1} dz +$$
$$+ \theta \left\{ (b_{42} + b_{43}) (\log x)^{n-1} \frac{(l_2 x)^{\nu+1}}{(\nu + 1)\,!} \nu^2 \right\}$$
$$= (\log x)^n I_{\nu-1,n} + \theta \left\{ (b_{42} + b_{43}) (\log x)^{n-1} \frac{(l_2 x)^{\nu+1}}{(\nu + 1)\,!} \nu^2 \right\}, \quad (127)$$

as in (126).

From (137) we get for $l_2 x \geqslant \nu \geqslant 2$; $1 \leqslant n \leqslant 10\, l_2 x$,

$$\sum_{r=0}^{\nu-1} B_r\, T(\nu - r - 1, n) = (\log x)^n \sum_{r=0}^{\nu-1} B_r\, I_{\nu-r-1,n} +$$
$$+ \theta \left\{ 2 (b_{42} + b_{43}) (\log x)^{n-1} \frac{(l_2 x)^{\nu+1}}{(\nu + 1)\,!} \nu^2 \sum_{r=0}^{\infty} | B_r | \right\}$$
$$= (\log x)^n \sum_{r=0}^{\nu-1} B_r\, I_{\nu-r-1,n} + \theta \left\{ b_{44} (\log x)^{n-1} \frac{(l_2 x)^{\nu+1}}{(\nu + 1)\,!} \nu^2 \right\}. \quad (138)$$

Using (110′), writing $t + 1$ for t' and taking $b_{38} \geqslant b_{44}$ the lemma follows immediately from (138).

Lemma 23. *If* $2 \leqslant \nu \leqslant l_2 x$, $1 \leqslant n \leqslant 10\, l_2 x$, *then*

$$\sum_{l=1}^{n-1} \binom{n}{l} \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} \frac{(-1)^{t-1}}{(n-l)^t} T(\nu - r - t - 1, l) +$$

$$+ \sum_{r=0}^{\nu-1} B_r\, T(\nu - r - 1, n)$$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu - r - t - 1)!} \times$$

$$\times \sum_{l=1}^{n-1} \sum_{s=1}^{l} \frac{(-1)^{s-1}}{s} \binom{l-1}{s-1} \binom{n}{l} \left\{ \frac{1}{(n-l)^t} - \frac{1}{(n-l+s)^t} \right\} +$$

$$+ (\log x)^n \sum_{r=0}^{r-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu - r - l - 1)!} \times$$

$$\times \sum_{l=1}^{n-1} \sum_{s=1}^{l} \frac{(-1)^{s}}{s} \binom{l-1}{s-1} \binom{n}{l} \left\{ \frac{1}{(n-l)^t} - \frac{1}{(n-l+s)^t} \right\} a^{n-l+s} +$$

$$+ (\log x)^n \sum_{r=0}^{\nu-1} B_r \sum_{t=0}^{\nu-r-1} (-1)^t \frac{(l_2 x)^{\nu-r-t-1}}{(\nu - r - t - 1)!} \times$$

$$\times \sum_{s=1}^{n} \frac{(-1)^{s-1}}{s} \binom{n-1}{s-1} \frac{1}{s^t} +$$

$$+ (\log x)^n \sum_{r=0}^{\nu-1} B_r \sum_{t=0}^{\nu-r-1} (-1)^t \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu - r - t - 1)!} \times$$

$$\times \sum_{s=1}^{n} \frac{(-1)^{s}}{s} \binom{n-1}{s-1} \frac{a^s}{s^t} +$$

$$+ \theta \left\{ b_{45} (\log x)^{n-1} g(\nu - 1, x) \frac{l_2 x}{\nu} \right\}. \qquad (139)$$

Proof. Follows immediately from Lemmas 21 and 22 since

$$\frac{(l_2 x)^\nu}{\nu!} \nu^3 \leqslant b_{46}\, g(\nu - 1, x), \qquad \text{by (102).}$$

LEMMA 24. *Under the conditions* (103),

$$\sum_{l=1}^{n}\binom{n}{l}\sum_{q_{\nu-1}\leqslant x^{a}}\frac{(\log q_{\nu-1})^{n-l}}{q_{\nu-1}}S(x/q_{\nu-1},l)$$

$$=(\log x)^{n}\sum_{r=0}^{\nu-2}B_{r}\sum_{t=1}^{\nu-r-1}(-1)^{t-1}\frac{(l_{2}x^{a})^{\nu-r-t-1}}{(\nu-r-t-1)!}\times$$

$$\times\sum_{l=1}^{n-1}\sum_{s=1}^{l}(-1)^{s}\frac{\binom{l-1}{s-1}\binom{n}{l}a^{n-l+s}}{s(n-l+s)^{t}}$$

$$+(\log x)^{n}\sum_{r=0}^{\nu-2}B_{r}\sum_{t=1}^{\nu-r-1}(-1)^{t-1}\frac{(l_{2}x^{a})^{\nu-r-t-1}}{(\nu-r-t-1)!}\times$$

$$\times\sum_{s=1}^{n}\frac{a^{s}}{s^{t}}\cdot\frac{(-1)^{s}}{s}\binom{n-1}{s-1}+$$

$$+(\log x)^{n}\sum_{r=0}^{\nu-2}B_{r}\sum_{t=1}^{\nu-r-1}(-1)^{t-1}\frac{(l_{2}x^{a})^{\nu-r-t-1}}{(\nu-r-t-1)!}\sum_{l=1}^{n-1}\frac{1}{l}\binom{n}{l}\frac{a^{n-l}}{(n-l)^{t}}+$$

$$+\frac{(\log x)^{n}}{n}\sum_{r=0}^{\nu-1}B_{r}\frac{(l_{2}x^{a})^{\nu-r-1}}{(\nu-r-1)!}+$$

$$+\theta\left\{b_{47}(\log x)^{n-1}g(\nu-1,x)\right\}.\qquad(140)$$

PROOF. By virtue of Lemma 14,

$$S(x/q_{\nu-1},l)=\frac{(\log \overline{x/q_{\nu-1}})^{l}}{l}+\theta\left\{b_{3}(\log\overline{x/q_{\nu-1}})^{l-1}\right\},$$

if $x/q_{\nu-1}\geqslant 2$ and $l\geqslant 1$. Hence

$$\sum_{l=1}^{n}\binom{n}{l}\sum_{q_{\nu-1}\leqslant x^{a}}\frac{(\log q_{\nu-1})^{n-l}}{q_{\nu-1}}S(x/q_{\nu-1},l)$$

$$=\sum_{l=1}^{n}\frac{\binom{n}{l}}{l}\sum_{q_{\nu-1}\leqslant x^{a}}\frac{(\log q_{\nu-1})^{n-l}}{q_{\nu-1}}(\log\overline{x/q_{\nu-1}})^{l}+$$

$$+\theta\left\{b_{3}\sum_{l=1}^{n}\binom{n}{l}\sum_{q_{\nu-1}\leqslant x^{a}}\frac{(\log q_{\nu-1})^{n-l}}{q_{\nu-1}}\cdot\frac{(\log\overline{x/q_{\nu-1}})^{l}}{(\log\overline{x/q_{\nu-1}}}\right\}.\qquad(141)$$

The modulus of the error term in (141) is not greater than

$$\frac{2\,b_3}{\log x}\sum_{q_{\nu-1}\leqslant x^{\alpha}}\sum_{l=0}^{n}\frac{(\log q_{\nu-1})^{n-l}\,(\log \overline{x/q_{\nu-1}})^{l}\binom{n}{l}}{q_{\nu-1}}$$

$$= 2\,b_3\,(\log x)^{n-1}\sum_{q_{\nu-1}\leqslant x^{\alpha}}\frac{1}{q_{\nu-1}},\qquad \text{by the bionomial theorem,}$$

$$\leqslant b_{48}\,(\log x)^{n-1}\frac{(l_2\,x)^{\nu-1}\,\nu}{(\nu-1)\,!},\qquad \text{by Lemma 15,}$$

$$\leqslant b_{48}\,(\log x)^{n-1}\frac{(l_2\,x)^{\nu}\,\nu}{\nu\,!},\qquad \text{since } l_2\,x \geqslant \nu,$$

$$\leqslant b_{49}\,(\log x)^{n-1}\,g\,(\nu-1, x),\qquad \text{by (102).} \tag{142}$$

Also

$$\sum_{l=1}^{n}\frac{1}{l}\binom{n}{l}\sum_{q_{\nu-1}\leqslant x^{\alpha}}\frac{(\log q_{\nu-1})^{n-l}\,(\log \overline{x/q_{\nu-1}})^{l}}{q_{\nu-1}}$$

$$=\sum_{l=1}^{n}\frac{1}{l}\binom{n}{l}\sum_{s=0}^{l}(-1)^{s}\binom{l}{s}(\log x)^{l-s}\,S_{\nu-1}\,(x^{\alpha}, n-l+s)$$

$$=\sum_{l=1}^{n-1}\sum_{s=1}^{l}\frac{(-1)^{s}}{s}\binom{l-1}{s-1}\binom{n}{l}(\log x)^{l-s}\,S_{\nu-1}\,(x^{\alpha}, n-l+s)+$$

$$+\sum_{s=1}^{n}\frac{(-1)^{s}}{s}\binom{n-1}{s-1}(\log x)^{n-s}\,S_{\nu-1}\,(x^{\alpha}, s)+$$

$$+\sum_{l=1}^{n-1}\frac{1}{l}\binom{n}{l}(\log x)^{l}\,S_{\nu-1}\,(x^{\alpha}, n-l)+\frac{1}{n}(\log x)^{n}\,S_{\nu-1}\,(x^{\alpha}, 0),$$

$$,\text{since }\ \frac{1}{l}\binom{l}{s}=\frac{1}{s}\binom{l-1}{s-1}\ \text{for}\ l\geqslant s\geqslant 1\,, \tag{143}$$

$$=\sum_{l=1}^{n-1}\sum_{s=1}^{l}\frac{(-1)^{s}}{s}\binom{l-1}{s-1}\binom{n}{l}(\log x)^{n}\times$$

$$\times\sum_{r=0}^{\nu-2}B_{r}\sum_{t=1}^{\nu-r-1}(-1)^{t-1}\frac{(l_2\,x^{\alpha})^{\nu-r-t-1}}{(\nu-r-t-1)\,!}\,\frac{\alpha^{n-l+s}}{(n-l+s)^{t}}+$$

$$+\sum_{s=1}^{n}\frac{(-1)^s}{s}\binom{n-1}{s-1}(\log x)^n\sum_{r=0}^{\nu-2}B_r\times$$

$$\times\sum_{t=1}^{\nu-r-1}(-1)^{t-1}\frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!}\frac{a^s}{s^t}+$$

$$+\sum_{l=1}^{n-1}\frac{1}{l}\binom{n}{l}(\log x)^n\sum_{r=0}^{\nu-2}B_r\times$$

$$\times\sum_{t=1}^{\nu-r-1}(-1)^{t-1}\frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!}\frac{a^{n-l}}{(n-l)^t}+$$

$$+\frac{(\log x)^n}{n}\sum_{r=0}^{\nu-1}B_r\frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!}+$$

$$+\sum_{l=1}^{n}\sum_{s=0}^{l}\frac{(-1)^s}{l}\binom{l}{s}\binom{n}{l}E_{\nu-1}(x^a, n-l+s)(\log x)^{l-s}, \quad (144)$$

by virtue of (84) and (36). Since the summations in (144) w.r.t. l and s can be interchanged w.r.t. r and t, we have only to show that

$$\left|\sum_{l=1}^{n}\sum_{s=0}^{l}\frac{(-1)^s}{l}\binom{l}{s}\binom{n}{l}E_{\nu-1}(x^a, n-l+s)(\log x)^{l-s}\right|$$

$$\leqslant b_{50}(\log x)^{n-1}g(\nu-1, x). \quad (145)$$

The lemma follows immediately from (145) on taking $b_{47}\geqslant b_{49}+b_{50}$, by virtue of (141), (142) and (144). Now

$$\sum_{l=1}^{n}\sum_{s=0}^{l}\frac{(-1)^s}{l}\binom{l}{s}\binom{n}{l}E_{\nu-1}(x^a, n-l+s)(\log x)^{l-s}$$

$$=E_{\nu-1}(x^a, n)\sum_{l=1}^{n}\frac{(-1)^l}{l}\binom{n}{l}+$$

$$+\sum_{\lambda=1}^{n}E_{\nu-1}(x^a, n-\lambda)(\log x)^\lambda\sum_{l=\lambda}^{n}\frac{(-1)^{l-\lambda}}{l}\binom{l}{l-\lambda}\binom{n}{l}. \quad (145\text{ a})$$

Also, if $\lambda \geqslant 1$,

$$\sum_{l=\lambda}^{n} \frac{(-1)^{l-\lambda}}{l} \binom{l}{l-\lambda} \binom{n}{l} = (-1)^{\lambda} \sum_{l=\lambda}^{n} \frac{(-1)^{l}}{l} \binom{l}{\lambda} \binom{n}{l}$$

$$= \frac{(-1)^{\lambda}}{\lambda} \sum_{l=\lambda}^{n} (-1)^{l} \binom{l-1}{\lambda-1} \binom{n}{l} = \frac{1}{\lambda}, \qquad (145\text{ b})$$

since, if $n \geqslant \lambda \geqslant 1$,

$$\sum_{l=\lambda}^{n} (-1)^{l} \binom{l-1}{\lambda-1} \binom{n}{l} = (-1)^{\lambda}. \qquad (145\text{ c})$$

[For, $\frac{1-x^n}{1-x} = 1 + x + \ldots + x^{n-1} = \left\{\frac{1-(1-\overline{1-x})^n}{1-x}\right\}$

$$= -\sum_{l=1}^{n} (-1)^{l} \binom{n}{l} (1-x)^{l-1}$$

$$= \sum_{l=1}^{n} (-1)^{l} \binom{n}{l} \sum_{\lambda=1}^{l} (-1)^{\lambda-1} \binom{l-1}{\lambda-1} x^{\lambda-1}$$

$$= -\sum_{\lambda=1}^{n} (-1)^{\lambda-1} x^{\lambda-1} \sum_{l=\lambda}^{n} (-1)^{l} \binom{l-1}{\lambda-1} \binom{n}{l},$$

and (145 c) follows from this on equating the coefficients of $x^{\lambda-1}$ on both sides]. From (145 b) we get

$$\sum_{\lambda=1}^{n} \frac{1}{\lambda} = \sum_{\lambda=1}^{n} \sum_{l=\lambda}^{n} \frac{(-1)^{l+\lambda}}{l} \binom{l}{\lambda} \binom{n}{l}$$

$$= \sum_{l=1}^{n} \frac{(-1)^{l}}{l} \binom{n}{l} \sum_{\lambda=1}^{l} (-1)^{\lambda} \binom{l}{\lambda} = -\sum_{l=1}^{n} \frac{(-1)^{l}}{l} \binom{n}{l}.$$

Hence

$$\sum_{l=1}^{n} \frac{(-1)^{l-1}}{l} \binom{n}{l} = \sum_{\lambda=1}^{n} \frac{1}{\lambda}. \qquad (146)$$

From (145 a), (145 b) and (146) we get

$$\left|\sum_{l=1}^{n}\sum_{s=0}^{l}\frac{(-1)^s}{l}\binom{l}{s}\binom{n}{l}E_{\nu-1}(x^a, n-l+s)(\log x)^{l-s}\right|$$

$$\leqslant |E_{\nu-1}(x^a, n)| \sum_{\lambda=1}^{n}\frac{1}{\lambda} + \sum_{\lambda=1}^{n}|E_{\nu-1}(x^a, n-\lambda)|(\log x)^{\lambda}$$

$$\leqslant n|E_{\nu-1}(x^a, n)| + \sum_{\lambda=1}^{n}|E_{\nu-1}(x^a, n-\lambda)|(\log x)^{\lambda}$$

$$\leqslant b\,n\,(\log x^a)^{n-1} g(\nu-1, x) + \sum_{\lambda=1}^{n} b(\log x^a)^{n-\lambda-1} g(\nu-1, x)(\log x)^{\lambda},$$

putting $u = x^{1-a}$ in the equation (106). Since $a = \frac{1}{2}$ this is not greater than

$$b(\log x)^{n-1} g(\nu-1, x)\left\{\frac{n}{2^{n-1}} + \sum_{\lambda=0}^{\infty}\frac{1}{2^{\lambda-1}}\right\} \leqslant 5\,b(\log x)^{n-1} g(\nu-1, x)$$

$$= b_{51}(\log x)^{n-1} g(\nu-1, x).$$

Hence (145) follows by taking $b_{50} \geqslant b_{51} = 5b$ where the constant b has been already chosen by virtue of (86).

Lemma 25. *Under the conditions* (103)

$$\sum_{l=1}^{n}\binom{n}{l} S(x^{1-a}, l)\, S_{\nu-1}(x^a, n-l)$$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1}\frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!} \times$$

$$\times \sum_{l=1}^{n-1}\sum_{s=1}^{l}\frac{(-1)^s}{s}\binom{l-1}{s-1}\binom{n}{l}\frac{a^{n-l+s}}{(n-l)^t} +$$

$$+ (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1}(-1)^{t-1}\frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!}\sum_{l=1}^{n-1}\frac{1}{l}\binom{n}{l}\frac{a^{n-l}}{(n-l)^t} +$$

$$+ (\log x)^n \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!} \sum_{s=1}^{n} \frac{(-1)^s}{s} \binom{n-1}{s-1} a^s +$$

$$+ \frac{(\log x)^n}{n} \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!} +$$

$$+ \theta \left\{ b_{52} (\log x)^{n-1} g(\nu-1, x) \right\}. \tag{147}$$

Proof. By virtue of Lemma 14 and (84), we have, for $1 \leqslant l < n-1$,

$$S(x^{1-a}, l).\ S_{\nu-1}(x^a, n-l)$$

$$= \left[\frac{(\log x)^l}{l} (1-a)^l + \theta \left\{ b_3 (\log x)^{l-1} (1-a)^{l-1} \right\} \right] \times$$

$$\times \left[(\log x)^{n-l} a^{n-l} \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!} \frac{1}{(n-l)^t} + \right.$$

$$\left. + E_{\nu-1}(x^a, n-l) \right]. \tag{148}$$

Now

$$\left| (\log x)^{l-1} (1-a)^{l-1} (\log x)^{n-l} a^{n-l} \sum_{r=0}^{\nu-2} B_r \times \right.$$

$$\left. \times \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!\,(n-l)^t} \right|$$

$$\leqslant 2 (\log x)^{n-1} (1-a)^l a^{n-l} \frac{(l_2 x)^\nu}{\nu!} \nu \sum_{r=0}^{\infty} |B_r|, \qquad \text{since } l_2 x \geqslant \nu,$$

$$= b_{53} (\log x)^{n-1} (1-a)^l a^{n-l}.\ \frac{(l_2 x)^\nu}{\nu!} \nu. \tag{149}$$

Also

$$\left\{ \frac{(\log x)^l}{l} (1-a)^l + b_3 (\log x)^{l-1} (1-a)^{l-1} \right\} | E_{\nu-1}(x^a, n-l) |$$

$$\leqslant (1 + 2 b_3) (\log x)^l (1-a)^l | E_{\nu-1}(x^a, n-l) |$$

$$\leqslant (1 + 2 b_3) (\log x)^l (1-a)^l b (\log x^a)^{n-l-1} g(\nu-1, x),$$

by putting $u = x^{1-a}$ in (106),

$$< b_{54} (\log x)^{n-1} (1-a)^l a^{n-l} g(\nu-1, x). \tag{150}$$

Hence for $1 \leqslant l \leqslant n-1$ we get from (148)-(150)

$$S(x^{1-a}, l)\, S_{\nu-1}(x^a, n-l)$$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!} \frac{(1-a)^l a^{n-1}}{l(n-l)^t} +$$

$$+ \theta \left\{ b_{55} (\log x)^{n-1} g(\nu-1, x)\, (1-a)^l a^{n-l} \right\}. \quad (151)$$

Also when $l = n$ we have by (36) and Lemma 14,

$$S(x^{1-a}, n)\, S_{\nu-1}(x^a, 0)$$

$$= \left[\frac{(\log x)^n (1-a)^n}{n} + \theta \left\{ b_3 (\log x)^{n-1} (1-a)^{n-1} \right\} \right] \times$$

$$\times \left[\sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!} + E_{\nu-1}(x^a, 0) \right]$$

$$= \left[\frac{(\log x)^n (1-a)^n}{n} \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!} + \right.$$

$$+ \theta \left\{ (1 + 2 b_3) (\log x)^n E_{\nu-1}(x^a, 0) \right\} +$$

$$+ \theta \left\{ 2 b_3 (\log x)^{n-1} (1-a)^n \frac{(l_2 x)^\nu}{\nu!} \sum_{r=0}^{\infty} |B_r| \right\}, \text{ since } \nu \leqslant l_2 x,$$

$$= \frac{(\log x)^n}{n} \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!} \sum_{s=0}^{n} (-1)^s \binom{n}{s} a^s +$$

$$+ \theta \left\{ b_{56} (\log x)^{n-1} g(\nu-1, x) \right\}, \text{ by virtue of (106),}$$

$$= (\log x)^n \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!} \sum_{s=1}^{n} \frac{(-1)^s}{s} \binom{n-1}{s-1} a^s +$$

$$+ \frac{(\log x)^n}{n} \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!} +$$

$$+ \theta \left\{ b_{56} (\log x)^{n-1} g(\nu-1, x) \right\}. \quad (152)$$

Hence from (151) and (152) we get

$$\sum_{l=1}^{n} \binom{n}{l} S(x^{1-a}, l)\, S_{\nu-1}(x^a, n-l)$$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!} \times$$

$$\times \sum_{l=1}^{n-1} \binom{n}{l} \frac{a^{n-l}}{(n-l)^t} \cdot \frac{(1-a)^l}{l}$$

$$+ (\log x)^n \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!} \sum_{s=1}^{n} \frac{(-1)^s}{s} \binom{n-1}{s-1} a^s +$$

$$+ \frac{(\log x)^n}{n} \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x^a)^{\nu-r-1}}{(\nu-r-1)!} +$$

$$+ \theta \left\{b_{55} (\log x)^{n-1} g(\nu-1, x) \sum_{l=1}^{n-1} \binom{n}{l} (1-a)^l a^{n-l}\right\}$$

$$+ \theta \left\{b_{56} (\log x)^{n-1} g(\nu-1, x)\right\}. \tag{153}$$

From (153) the lemma follows immediately on noting that

$$\frac{(1-a)^l}{l} = \frac{1}{l} + \sum_{s=1}^{l} \frac{(-1)^s}{l} \binom{l}{s} a^s = \frac{1}{l} + \sum_{s=1}^{l} \frac{(-1)^s}{s} \binom{l-1}{s-1} a^s,$$

$$\sum_{l=1}^{n-1} \binom{n}{l} (1-a)\, a^{n-l} \leqslant \sum_{l=0}^{n} \binom{n}{l} (1-a)^l a^{n-l} = 1.$$

We have only to choose

$$b_{52} > b_{55} + b_{56}.$$

LEMMA 26. *Under the conditions of* (103),

$$\sum_{l=0}^{n} \binom{n}{l} \Big\{ \sum_{p \leqslant x^{1-a}} \left(\frac{\log p}{p}\right)^l S_{\nu-1}(x/p, n-l) +$$

$$+ \sum_{q_{\nu-1} \leqslant x^a} \frac{(\log q_{\nu-1})^{n-l}}{q_{\nu-1}} S(x/q_{\nu-1}, l) - S(x^{1-a}, l) . S_{\nu-1}(x^a, n-l) \Big\}$$

$$= \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} \frac{(-1)^{t-1}}{n^t} \sum_{p<x^{1-a}} \frac{(\log x - \log p)^n}{p} \frac{(l_2\, \overline{x/p})^{\nu-r-t-1}}{(\nu-r-t-1)\,!} +$$

$$+ \sum_{q_{\nu-1}<x^a} \frac{(\log q_{\nu-1})^n}{q_{\nu-1}} S\,(x/q_{\nu-1}) - S\,(x^{1-a}).\quad S_{\nu-1}\,(x^a, n) +$$

$$+ (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2\, x)^{\nu-r-t-1}}{(\nu-r-t-1)\,!} \times$$

$$\times \sum_{l=1}^{n-1} \sum_{s=1}^{l} \frac{(-1)^{s-1}}{s} \binom{l-1}{s-1} \binom{n}{l} \left\{ \frac{1}{(n-l)^t} - \frac{1}{(n-l+s)^t} \right\} +$$

$$+ (\log x)^n \sum_{r=0}^{\nu-1} B_r \sum_{t=0}^{\nu-r-1} (-1)^t \frac{(l_2\, x)^{\nu-r-t-1}}{(\nu-r-t-1)\,!} \times$$

$$\times \sum_{s=1}^{n} \frac{(-1)^{s-1}}{s} \binom{n-1}{s-1} \frac{1}{s^t} +$$

$$+ \theta \left\{ b\,(\log x)^{n-1}\, g\,(\nu-1, x)\; l_2\, x \left(1 + \frac{b_{57}}{\nu} \right) \right\}. \tag{154}$$

Proof. We first use Lemma 19 and then substitute on the left side of (154) the contribution made by $1 \leqslant l \leqslant n$ by using Lemmas 23, 24 and 25.

As to the dominant expression we have only to show that the terms containing a in the Lemmas 23-25 cancel out. The terms containing a in Lemma 23 cancel with the first two lines of (140) and 1st and 3rd lines of (147). Also the 3rd and 4th lines of (140) cancel with the 2nd and 4th lines of (147).

Also the modulus of the error term does not exceed (since $b > 1$ by (86))

$$b\,(\log x)^{n-1}\, g\,(\nu-1, x)\; l_2\, x \left\{ 1 + \frac{b_{17}}{\nu} + \frac{b_{45}}{\nu} + \frac{b_{47}}{\nu} + \frac{b_{52}}{\nu} \right\},$$

since $\nu < \frac{1}{2}\, x$,

$$= b\,(\log x)^{n-1}\, g\,(\nu-1, x)\; l_2\, x \left\{ 1 + \frac{b_{17} + b_{45} + b_{47} + b_{52}}{\nu} \right\}$$

$$< b\,(\log x)^{n-1}\,g(\nu-1,x)\,l_2\,x\left(1+\frac{b_{57}}{\nu}\right),$$

on taking $b_{57} > b_{17}+b_{45}+b_{47}+b_{52}$. Hence the lemma.

LEMMA 27. *Under the conditions of* (103),

$$\sum_{q_{\nu-1}\leqslant x^{a}} \frac{(\log q_{\nu-1})^n}{q_{\nu-1}}\, S(x/q_{\nu-1})$$

$$= (l_2\,x+B)\,S_{\nu-1}(x^a,n) -$$

$$-(\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1}\, \frac{(l_2\,x^a)^{\nu-r-t-1}}{(\nu-r-t-1)\,!} \sum_{\mu=1}^{\infty} \frac{a^{n+\mu}}{\mu\,(n+\mu)t} +$$

$$+\,\theta\,\{\,b_{58}\,(\log x)^{n-1}\,g(\nu-1,x)\,\}. \tag{155}$$

PROOF. It is well known that for $x > 2$,

$$S(x) = \sum_{p<x}\frac{1}{p} = l_2\,x + B + \theta\left(\frac{b_{59}}{\log x}\right).$$

Hence

$$\sum_{q_{\nu-1}\leqslant x^{a}} \frac{(\log q_{\nu-1})^n}{q_{\nu-1}}\, S(x/q_{\nu-1})$$

$$= \sum_{q_{\nu-1}\leqslant x^{a}} \frac{(\log q_{\nu-1})^n}{q_{\nu-1}} \left\{ l_2\,\overline{x/q_{\nu-1}} + B + \theta\left(\frac{b_{60}}{\log x}\right)\right\}$$

$$= (l_2\,x+B)\,S_{\nu-1}(x^a,n) + \sum_{q_{\nu-1}\leqslant x^{a}} \frac{(\log q_{\nu-1})^n}{q_{\nu-1}} \log\left(1-\frac{\log q_{\nu-1}}{\log x}\right) +$$

$$+\,\theta\left\{\frac{b_{60}}{\log x}\,S_{\nu-1}(x^a,n)\right\}. \tag{156}$$

Now, by Lemma 15,

$$\frac{b_{60}}{\log x}\,S_{\nu-1}(x^a,n) < \frac{b_{61}}{\log x}\cdot\frac{(l_2\,x+b_8)^{\nu-1}}{(\nu-1)\,!}\,\nu(\log x)^n$$

$$< b_{62}\,\frac{(l_2\,x)^{\nu}}{\nu\,!}\,(\log x)^{n-1}\,\nu$$

$$< b_{64}\,(\log x)^{n-1}\,g\,(\nu-1,x), \tag{157}$$

since $\nu < l_2\,x$ and because of (102).

We shall now show that for $1 < n < 10\,(l_2 x - \nu \log 2)$,

$$- \sum_{q_{\nu-1} \leqslant x^{\alpha}} \frac{(\log q_{\nu-1})^n}{q_{\nu-1}} \log\left(1 - \frac{\log q_{\nu-1}}{\log x}\right)$$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^{\alpha})^{\nu-r-t-1}}{(\nu-r-t-1)\,!} \sum_{\mu=1}^{\infty} \frac{\alpha^{n+\mu}}{\mu\,(n+\mu)^t} +$$

$$+ \theta\left\{ b_{65} (\log x)^{n-1} g(\nu-1, x) \right\}. \quad (158)$$

The lemma will then follow immediately from (157)-(158) by taking

$$b_{58} > b_{64} + b_{65}\,.$$

Let

$$s' = [\,10\,(l_2 x - \nu \log 2)\,]. \quad (159)$$

Then

$$1 < 2\,l_2 x - 1 < s' < 10\,l_2 x. \quad (160)$$

When $n = s'$,

$$\left| \sum_{q_{\nu-1} \leqslant x^{\alpha}} \frac{(\log q_{\nu-1})^n}{q_{\nu-1}} \log\left(1 - \frac{\log q_{\nu-1}}{\log x}\right) \right|$$

$$< b_{66} \sum_{q_{\nu-1} \leqslant x^{\alpha}} \frac{(\log q_{\nu-1})^{s'}}{q_{\nu-1}}$$

$$< b_{67} \frac{(l_2 x + b_8)^{\nu-1}}{(\nu-1)\,!} \,\nu\, (\log x)^{s'} \alpha^{s'}, \quad \text{by Lemma 15,}$$

$$< b_{68} \frac{(l_2 x)^{\nu}}{\nu\,!} \,\nu\, (\log x)^{s'-1},$$

$$\text{since } \alpha^{s'} = \frac{1}{2^{s'}} < \frac{1}{2^{l_2 x - 1}} < \frac{2}{\log x}, \quad (161)$$

$$< b_{69} (\log x)^{s'-1} g(\nu-1, x), \quad (162)$$

by virtue of (102). Also

$$\left| (\log x)^{s'} \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^{\alpha})^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{\mu=1}^{\infty} \frac{\alpha^{s'+\mu}}{\mu\,(s'+\mu)^t} \right|$$

$$< (\log x)^{s'} \sum_{r=0}^{\nu-2} |\,B_r\,| \frac{(l_2 x)^{\nu}}{\nu\,!} \,\nu \sum_{\mu=1}^{\infty} \alpha^{s'+\mu}$$

$$< (\log x)^{s'} \frac{(l_2 x)^{\nu}}{\nu\,!} \nu\, a^{s'} \sum_{r=0}^{\infty} |B_r|$$

$$< b_{70} (\log x)^{s'-1} g(\nu - 1, x), \tag{163}$$

by virtue of (161) and (102). Hence (158) holds when $n = s'$ if we choose

$$b_{65} > b_{69} + b_{70}. \tag{164}$$

Let now

$$1 < n < s' - 1. \tag{165}$$

Then, for $q_{\nu-1} < x^{a}$,

$$-\log\left(1 - \frac{\log q_{\nu-1}}{\log x}\right) = \sum_{s=1}^{s'-n} \frac{1}{s} \frac{(\log q_{\nu-1})^{s}}{(\log x)^{s}} + \theta\left\{ b_{71} \frac{(\log q_{\nu-1})^{s'-n+1}}{(\log x)^{s'-n-1}} \right\}.$$

Hence, if $1 \leqslant n \leqslant s' - 1$,

$$-\sum_{q_{\nu-1} < x^{a}} \frac{(\log q_{\nu-1})^{n}}{q_{\nu-1}} \log\left(1 - \frac{\log q_{\nu-1}}{\log x}\right)$$

$$= \sum_{s=1}^{s'-n} \frac{1}{s(\log x)^{s}} S_{\nu-1}(x^{a}, n+s) +$$

$$+ \theta\left\{ b_{71} \frac{1}{(\log x)^{s'-n+1}} \cdot S_{\nu-1}(x^{a}, s'+1) \right\}. \tag{166}$$

Now, by virtue of (84) and (106), for

$$1 < n + s \leqslant s' = [10(l_2 x - \nu \log 2)],$$

we get

$$S_{\nu-1}(x^{a}, n+s)$$

$$= (\log x)^{n+s} a^{n+s} \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^{a})^{\nu-r-t-1}}{(\nu-r-t-1)\,!} \frac{1}{(n+s)^{t}} +$$

$$+ \theta\left\{ b (\log x)^{n+s-1} a^{n+s-1} g(\nu - 1, x) \right\}. \tag{167}$$

Hence, for $1 \leqslant n \leqslant s' - 1$,

$$\sum_{s=1}^{s'-n} \frac{1}{s\,(\log x)^{s}} S_{\nu-1}(x^{a}, n+s)$$

$$= (\log x)^{n} \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^{a})^{\nu-r-t-1}}{(\nu-r-t-1)\,!} \sum_{s=1}^{s'-n} \frac{a^{n+s}}{s\,(n+s)^{t}} +$$

$$+ \theta\left\{ b\,(\log x)^{n-1} g(\nu - 1, x) \sum_{s=1}^{\infty} a^{n+s-1} \right\}$$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{s=1}^{\infty} \frac{a^{n+s}}{s\,(n+s)^t} -$$

$$- (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{s=s'-n+1}^{\infty} \frac{a^{n+s}}{s\,(n+s)^t} +$$

$$+ \theta \left\{ b_{72} (\log x)^{n-1} g(\nu-1, x) \right\}, \text{ since } a^n \leqslant \tfrac{1}{2}$$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{\mu=1}^{\infty} \frac{a^{n+\mu}}{\mu\,(n+\mu)^t} +$$

$$+ \theta \left\{ b_{73} (\log x)^{n-1} g(\nu-1, x) \right\}, \qquad (168)$$

since

$$\left| (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{s=s'-n+1}^{\infty} \frac{a^{n+s}}{s\,(n+s)^t} \right|$$

$$\leqslant (\log x)^n \sum_{r=0}^{\nu-2} |B_r| \frac{(l_2 x)^\nu}{\nu!} \nu \,.\sum_{s=s'-n+1}^{\infty} \frac{1}{2^{n+s}}$$

$$\leqslant 2 (\log x)^n \frac{(l_2 x)^\nu}{\nu!} \nu \frac{1}{2^{s'+1}} \sum_{r=0}^{\infty} |B_r|$$

$$\leqslant b_{74} (\log x)^n \frac{(l_2 x)^\nu}{\nu!} \frac{\nu}{2^{l_2 x}} \leqslant b_{75} (\log x)^{n-1} g(\nu-1, x), \qquad (169)$$

by virtue of (102). Also

$$b_{71} \frac{1}{(\log x)^{s'-n+1}} S_{\nu-1}(x^a, s'+1)$$

$$\leqslant b_{76} \frac{1}{(\log x)^{s'-n+1}} (\log x)^{s'+1} a^{s'+1} \frac{(l_2 x + b_8)^{\nu-1}}{(\nu-1)!} \nu, \text{ by Lemma 15,}$$

$$\leqslant b_{77} (\log x)^n \frac{(l_2 x)^\nu}{\nu!} \nu \frac{1}{2^{2l_2 x}} \leqslant b_{78} (\log x)^{n-1} g(\nu-1, x),$$

by virtue of (102). (170)

Hence from (166), (168) and (170), for $1 \leqslant n \leqslant s'-1$,

$$- \sum_{q_{\nu-1} \leqslant x^a} \frac{(\log q_{\nu-1})^n}{q_{\nu-1}} \log \left(1 - \frac{\log q_{\nu-1}}{\log x}\right)$$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{\mu=1}^{\infty} \frac{a^{n+\mu}}{\mu(n+\mu)^t} +$$

$$+ \theta\{b_{79} (\log x)^{n-1} g(\nu-1, x)\}. \qquad (171)$$

Hence the lemma follows from (164) and (171) if $b_{65} \geqslant b_{69} + b_{70}$ + ..., since thereby (155) is proved for $1 \leqslant n \leqslant 10\,(l_2 x - \nu \log 2)$ under the conditions of (103).

LEMMA 28. *Under the conditions of* (103),

$$S(x^{1-a}) \cdot S_{\nu-1}(x^a, n) = (l_2 x + B)\, S_{\nu-1}(x^a, n) + (\log x)^n \log (1-a) \times \times \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)\,!} \frac{a^n}{n^t} + + \theta\{b_{80} (\log x)^{n-1} g(\nu - 1, x)\}. \qquad (172)$$

PROOF. We know that

$$S(x) = l_2 x + B + \theta\left(\frac{b_{81}}{\log x}\right).$$

Hence

$$S(x^{1-a}) = l_2 x + \log (1-a) + B + \theta\left(\frac{2\, b_{81}}{\log x}\right)$$

$$S(x^{1-a})\, S_{\nu-1}(x^a, n) = (l_2 x + B)\, S_{\nu-1}(x^a, n) + \log (1-a)\, S_{\nu-1}(x^a, n) + + \theta\left\{\frac{2\, b_{81}}{\log x} S_{\nu-1}(x^a, n)\right\}$$

$$= (l_2 x + B)\, S_{\nu-1}(x^a, n) + \log (1-a)\, (\log x)^n \times \times \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t-1}}{(\nu-r-t-1)\,!} \frac{a^n}{n^t} +$$

$$+ \theta\left\{|\log (1 - a|\, |E_{\nu-1}(x^a, n)| + \frac{2\, b_{81}}{\log x} S_{\nu-1}(x^a, n)\right\}. \qquad (173)$$

Also

$$|E_{\nu-1}(x^a, n)| < b\, g(\nu - 1, x)\, (\log x)^{n-1}, \qquad \text{by (106)},$$

and,

$$\frac{2\, b_{81}}{\log x} S_{\nu-1}(x^a, n) \leqslant \frac{b_{82}}{\log x} \frac{(l_2 x + b_8)^{\nu-1}}{(\nu-1)\,!} (\log x)^n, \qquad \text{by Lemma 15,}$$

$$\leqslant b_{83} (\log x)^{n-1} \frac{(l_2 x)^\nu}{\nu\,!}$$

$$\leqslant b_{83} (\log x)^{n-1} g(\nu - 1, x). \qquad (175)$$

From (173)-(175) the lemma follows by taking

$$b_{80} > b + b_{83}.$$

LEMMA 29. *In agreement with the notation of Lemma* 20, *let*

$$T(\nu-1) = T(\nu-1, 0, x^{1-\alpha})$$
$$= \sum_{p\leqslant x^{1-\alpha}} \frac{(\log x - \log p)^n}{p} \frac{(l_2 x/p)^{\nu-1}}{(\nu-1)!}. \quad (176)$$

Then for $1 \leqslant \nu \leqslant l_2 x$, $1 \leqslant n \leqslant 10\, l_2 x$,

$$T(\nu-1) = (\log x)^n \left\{ \frac{(l_2 x)^\nu}{(\nu-1)!} + B \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \right\} +$$
$$+ (\log x)^n \sum_{t'=1}^{\nu-1} (-1)^{t'} \frac{(l_2 x)^{\nu-t'-1}}{(\nu-t'-1)!} \sum_{\mu=1}^{\infty} \frac{1}{(n+\mu)^{t'+1}} -$$
$$- (\log x)^n \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} \sum_{\mu=1}^{\infty} \frac{n}{\mu(n+\mu)} +$$
$$+ (\log x)^n \sum_{t'=1}^{\nu-1} (-1)^{t'-1} \frac{(l_2 x^\alpha)^{\nu-t'-1}}{(\nu-t'-1)!} \sum_{\mu=1}^{\infty} \frac{\alpha^{n+\mu}}{(n+\mu)^{t'+1}} +$$
$$+ (\log x)^n \frac{(l_2 x^\alpha)^{\nu-1}}{(\nu-1)!} \sum_{\mu=1}^{\infty} \frac{n\, \alpha^{n+\mu}}{\mu(n+\mu)} +$$
$$+ \log(1-\alpha)\, (\log x)^n \frac{(l_2 x^\alpha)^{\nu-1}}{(\nu-1)!} \alpha^n +$$
$$+ \theta \left\{ b_{84} \frac{(l_2 x)^{\nu+2}}{(\nu+2)!} (\log x)^{n-1} \nu^8 \right\}. \quad (177)$$

Empty sums when $\nu = 1$ *are regarded as equal to* 0.

PROOF. Let $0 < \gamma < 1$ be an absolute constant satisfying both the equations

$$g(x) = \sum_{p\leqslant x} \log p = x + O\left(x/e^{\gamma\sqrt{(lx l_2 x)}}\right), \quad (178)$$

$$\sum_{p\leqslant x} \frac{1}{p} = l_2 x + B + O\left(\frac{1}{e^{\gamma\sqrt{(lx l_2 x)}}}\right). \quad (179)$$

Further let η be chosen such that

$$\log \eta \log\log \eta = \frac{4}{\gamma^2} (l_2 x)^2. \quad (180)$$

Then obviously $\log \eta \geqslant 2\, l_2 x$, or

$$\eta \geqslant (\log x)^2. \quad (181)$$

Now

$$T(\nu-1) = \sum_{p \leqslant x^{1-\alpha}} \frac{(\log x - \log p)^n}{p} \frac{(l_2 \overline{x/p})^{\nu-1}}{(\nu-1)!}$$

$$= \sum_{p \leqslant \eta} + \sum_{\eta < p \leqslant x^{1-\alpha}}$$

$$= \sum_{p \leqslant \eta} + \sum_{\eta \leqslant p \leqslant x^{1-\alpha}} + \theta\left\{(\log x)^{n-1} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!}\right\}, \text{by (181)},$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \theta\left\{\frac{(l_2 x)^{\nu-1}}{(\nu-1)!}(\log x)^{n-1}\right\}, \text{ say.} \qquad (182)$$

It may be remarked that (177) obviously holds for any finite value of x if b_{84} is properly chosen; hence in (182) we may suppose x to be so large that $\eta < x^{1-\alpha}$. Compare the equations (124′), (124″) and (125).

We consider $\sum_1$ first. For $p \leqslant \eta$,

$$(l_2 \overline{x/p})^{\nu-1} (\log x - \log p)^n$$

$$= (l_2 x)^{\nu-1} (\log x)^n \left\{1 + \frac{\log(1 - lp/lx)}{l_2 x}\right\}^{\nu-1} \left\{1 - \frac{lp}{lx}\right\}^n$$

$$= (l_2 x)^{\nu-1} (\log x)^n \left\{1 + \theta\left(b_{85} \frac{\log \eta}{\log x}\right)\right\} \times$$

$$\times \left\{1 + \theta\left(b_{86} \frac{l_2 x \log \eta}{\log x}\right)\right\}$$

$$= (l_2 x)^{\nu-1} (\log x)^n \left\{1 + \theta\left(b_{87} \frac{\log \eta\, l_2 x}{\log x}\right)\right\}. \qquad (182\text{A})$$

Hence

$$\sum\nolimits_1 = \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^n \left\{1 + \theta\left(b_{87} \frac{\log \eta . l_2 x}{\log x}\right)\right\} \times$$

$$\times \left\{l_2 \eta + B + \theta\left(b_{88} \frac{1}{e^{\gamma\sqrt{(l\eta/l_2\eta)}}}\right)\right\}, \text{ by virtue of (179).}$$

$$= \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^n \left\{1 + \theta\left(b_{87} \frac{\log \eta . l_2 x}{\log x}\right)\right\} \times$$

$$\times \left\{l_2 \eta + B + \theta\left(\frac{b_{88}}{\log x}\right)\right\}, \qquad \text{by (180),}$$

$$= \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^n \left\{ l_2\eta + B + \theta \left(b_{89} \frac{\log \eta . l_2\eta . l_2 x}{\log x} \right) \right\}$$

$$= \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^n (l_2\eta + B) + \theta \left\{ b_{90} \frac{(l_2 x)^{\nu+2}}{(\nu+2)!} (\log x)^{n-1} \nu^3 \right\}, \tag{183}$$

by (180). We now consider $\sum_2$.

$$F(u) = F(u, x) = \frac{(l_2 \overline{x/u})^{\nu-1}}{(\nu-1)!} \frac{(\log x - \log u)^n}{u \log u} \tag{184}$$

is obviously a decreasing function of u for $u \leqslant x^{1-a}$.

$$\sum_2 = \sum_{\eta \leqslant p \leqslant x^{1-a}} \frac{(l_2 \overline{x/p})^{\nu-1}}{(\nu-1)!} \frac{(\log x - \log p)^n}{p}$$

$$= \sum_{\eta \leqslant m \leqslant x^{1-a}} \frac{(l_2 \overline{x/m})^{\nu-1}}{(\nu-1)!} \frac{(\log x - \log m)^n}{m} \frac{\{ g(m) - g(m-1) \}}{\log m}$$

$$= \sum_{\eta \leqslant m \leqslant x^{1-a}} F(m, x) + \sum_{\eta \leqslant m \leqslant x^{1-a}} F(m, x) \{ \epsilon(m) - \epsilon(m-1) \},$$

where, as before,

$$g(x) = x + \epsilon(x), \tag{185}$$

so that by (178) we have

$$| \epsilon(x) | < \frac{b_{92} \cdot x}{e^{\gamma \sqrt{[lx l_2 x]}}}. \tag{186}$$

Hence

$$\left| \sum_{\eta \leqslant m \leqslant x^{1-a}} F(m) \{ \epsilon(m \, \epsilon(m-1) \} \right|$$

$$< \sum_{\eta \leqslant m \leqslant x^{1-a}} | \epsilon(m) | \{ F(m) - F(m+1) \}, +$$

$$+ | \epsilon([x^{1-a}]) | \; | F([x^{1-a}] + 1) |$$

$$+ F([\eta]) \{ | \epsilon([\eta] - 1) | + | \epsilon([\eta]) |$$

$$< b_{92} \sum_{\eta \leqslant m \leqslant x^{1-a}} \frac{m}{e^{\gamma\sqrt{(lm\, l_2 m)}}} \{ F(m) - F(m+1) \} +$$

$$+ b_{93} \frac{x^{1-a}}{e^{b_{94}\sqrt{(\log x)}}} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!} \frac{(\log x)^n}{x^{1-a} \log x} +$$

$$+ b_{93} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!} \frac{(\log x)^n}{\eta} \frac{\eta}{e^{\gamma\sqrt{[\log(\eta-2)\, l_2(\eta-2)]}}}, \qquad \text{by (186)},$$

$$< b_{92} \sum_{\eta \leqslant m \leqslant x^{1-a}} \frac{m}{e^{\gamma\sqrt{(lm\, l_2 m)}}} \{ F(m) - F(m+1) \} +$$

$$+ b_{94} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!} (\log x)^{n-1},$$

since $\dfrac{1}{e^{\gamma\sqrt{[\log(\eta-2)\, l_2(\eta-2)]}}} < \dfrac{b_{95}}{e^{(\gamma/2)\sqrt{[\log \eta\, l_2 \eta]}}} < \dfrac{b_{95}}{(\log x)}$, by (180).

Hence

$$\left| \sum_{\eta \leqslant m \leqslant x^{1-a}} F(m) \{ \epsilon(m) - \epsilon(m-1) \} \right|$$

$$< b_{96} \sum_{\eta \leqslant m \leqslant x^{1-a}} F(m) \left\{ \frac{m}{e^{\gamma\sqrt{(lm\, l_2 m)}}} - \frac{(m-1)}{e^{\gamma\sqrt{[l(m-1)\, l_2(m-1)]}}} \right\} +$$

$$+ b_{96} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!} (\log x)^{n-1}$$

$$\leqslant b_{97} \sum_{\eta \leqslant m \leqslant x^{1-a}} F(m)/e^{\gamma\sqrt{(lm\, l_2 m)}} + b_{97} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!} (\log x)^{n-1}$$

$$\leqslant b_{97} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!} (\log x)^n \sum_{m \geqslant \eta} \frac{1}{m \log m\; e^{\gamma\sqrt{(lm\, l_2 m)}}} +$$

$$+ b_{97} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!} (\log x)^{n-1}$$

$$\leqslant b_{98} \frac{(l_2 x)^{\nu-1}}{(\nu-1)\,!} (\log x)^{n-1}, \tag{187}$$

since

$$\sum_{m \geqslant \eta} \frac{1}{m \log m\; e^{\gamma\sqrt{(lm\, l_2 m)}}} < \frac{1}{e^{(\gamma/2)\sqrt{(l\eta\, l_2 \eta)}}} \sum_{m \geqslant \eta} \frac{1}{m \log m\; e^{(\gamma/2)\sqrt{lm}}}$$

$$\leqslant \frac{1}{\log x} \sum_{m=2}^{\infty} \frac{1}{m \log m\; e^{(\gamma/2)\sqrt{\log m}}}, \qquad \text{by (180)},$$

$$< b_{99}/\log x.$$

Hence from (185) and (187) we get

$$\sum\nolimits_2 = \sum_{\eta \leqslant m \leqslant x^{1-\alpha}} F(m, x) + \theta \left\{ b_{98} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^{n-1} \right\}$$

$$= \int_\eta^{x^{1-\alpha}} \frac{(l_2 x/u)^{\nu-1}}{(\nu-1)!} \frac{(\log x/u)^n}{u \log u} du + \theta \left\{ b_{100} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^{n-1} \right\},$$

by (181) and (184),

$$= \int_{x^\alpha}^{x/\eta} \frac{(l_2 v)^{\nu-1}}{(\nu-1)!} \frac{(\log v)^n}{v(l x - l v)} dv + \theta \left\{ b_{100} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^{n-1} \right\}$$

$$= \int_{\alpha l x}^{lx - l\eta} \frac{(\log w)^{\nu-1}}{(\nu-1)!} \frac{w^n}{\log x - w} dw + \theta \left\{ b_{100} \frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^{n-1} \right\};$$

(188)

$$I = \int_{\alpha l x}^{lx - l\eta} \frac{(\log w)^{\nu-1}}{(\nu-1)!} \frac{w^n}{\log x - w} dw$$

$$= \left[-\log(\log x - w) \frac{(\log w)^{\nu-1}}{(\nu-1)!} w^n \right]_{\alpha l x}^{lx - l\eta} +$$

$$+ \int_{\alpha l x}^{lx - l\eta} \{\log(\log x - w)\} \left\{ \frac{(\log w)^{\nu-2}}{(\nu-2)!} w^{n-1} + \right.$$

$$\left. + \frac{n(\log w)^{\nu-1}}{(\nu-1)!} w^{n-1} \right\} dw. \qquad (188)$$

Now

$$\{\log(l x - l \eta)\}^{\nu-1} \{\log x - \log \eta\}^n$$

$$= (l_2 x)^{\nu-1} (\log x)^n \left\{ 1 + \theta \left(b_{101} \frac{\log \eta \, l_2 x}{\log x} \right) \right\},$$

as in (182A). Hence, by (189),

$$I = -\frac{(l_2 x)^{\nu-1}}{(\nu-1)!} (\log x)^n l_2 \eta \left\{ 1 + \theta \left(b_{101} \frac{\log \eta \, l_2 x}{\log x} \right) \right\} +$$

$$+ \{ l_2 x + \log(1 - \alpha) \} \frac{(l_2 x^\alpha)^{\nu-1}}{(\nu-1)!} (\log x)^n \alpha^n +$$

$$+ \int_{\alpha l x}^{lx - l\eta} \{\log(\log x - w)\} \left\{ \frac{(\log w)^{\nu-2}}{(\nu-2)!} w^{n-1} + \right.$$

$$\left. + n \frac{(\log w)^{\nu-1}}{(\nu-1)!} w^{n-1} \right\} dw$$

$$= -\frac{(l_2 x)^{\nu-1}}{(\nu-1)!}(\log x)^n l_2\eta + \{l_2 x + \log(1-x)\} \times$$

$$\times \frac{(l_2 x^a)^{\nu-1}}{(\nu-1)!}(\log x)^n a^n$$

$$+ \int_{alx}^{lx-l\eta} \{\log(\log x - w)\} \left\{\frac{(\log w)^{\nu-2}}{(\nu-2)!} w^{n-1} + \right.$$

$$\left. + n\frac{(\log w)^{\nu-1}}{(\nu-1)!} w^{n-1}\right\} dw +$$

$$+ \theta\left\{b_{102}\frac{(l_2 x)^{\nu+2}}{(\nu+2)!}(\log x)^{n-1}\nu^3\right\}, \tag{190}$$

by (180). Now

$$\left|\int_{lx-\eta}^{lx}\{\log(\log x - w)\} w^{n-1}\left\{\frac{(\log w)^{\nu-2}}{(\nu-2)!} + n\frac{(\log w)^{\nu-1}}{(\nu-1)!}\right\} dw\right|$$

$$< 2\frac{(l_2 x)^{\nu-1}}{(\nu-1)!} n(\log x)^{n-1}\int_{lx-l\eta}^{lx}|\log(\log x - w)|\,dw$$

$$= 2\frac{(l_2 x)^{\nu-1}}{(\nu-1)!} n(\log x)^{n-1}\log\eta\int_0^1|\log(z\log\eta)|\,dz,$$

$$= 2\frac{(l_2 x)^{\nu-1}}{(\nu-1)!} n(\log x)^{n-1}\log\eta\left\{l_2\eta + \int_0^1|\log z|\,dz\right\}$$

$$< b_{103}\frac{(l_2 x)^{\nu+2}}{(\nu+2)!}(\log x)^{\nu-1}\nu^3,$$

by (180), since $\int_0^1 \log z\, dz$ is convergent.

Hence by (190) we get :

$$I = -\frac{(l_2 x)^{\nu-1}}{(\nu-1)!}(\log x)^n l_2\eta + \{l_2 x + \log(1-a)\}\frac{(l_2 x^a)^{\nu-1}}{(\nu-1)!}(\log x)^n a^n +$$

$$+ \int_{a\log x}^{\log x}\{\log(\log x - w)\} w^{n-1}\left\{\frac{(\log w)^{\nu-2}}{(\nu-2)!} + n\frac{(\log w)^{\nu-1}}{(\nu-1)!}\right\} dw +$$

$$+ \theta\left\{b_{104}\frac{(l_2 x)^{\nu+2}}{(\nu+2)!}(\log x)^{n-1}\nu^3\right\}$$

$$= -\frac{(l_2 x)^{\nu-1}}{(\nu-1)!}(\log x)^n l_2 \eta + \{ l_2 x + \log(1-a) \} \frac{(l_2 x^a)^{\nu-1}}{(\nu-1)!}(\log x)^n a^n +$$

$$+ l_2 x \left[\frac{(\log w)^{\nu-1}}{(\nu-1)!} w^n \right]_{a \log x}^{\log x} +$$

$$+ \int_{a \log x}^{\log x} \log\left(1 - \frac{w}{\log x}\right) \frac{(\log w)^{\nu-2}}{(\nu-2)!} w^{n-1}\, dw +$$

$$+ n \int_{a \log x}^{\log x} \log\left(1 - \frac{w}{\log x}\right) \frac{(\log w)^{\nu-1}}{(\nu-1)!} w^{n-1}\, dw +$$

$$+ \theta \left\{ b_{104} \frac{(l_2 x)^{\nu+2}}{(\nu+2)!} (\log x)^{n-1} \nu^3 \right\}. \tag{191}$$

From (182), (183), (188) and (191) we get, noting that $\nu \leqslant l_2 x$,

$$T(\nu-1) = B \frac{(l_2 x)^{\nu-1}}{(\nu-1)!}(\log x)^n + \frac{(l_2 x)^{\nu}}{(\nu-1)!}(\log x)^n +$$

$$+ \log(1-a) \frac{(l_2 x^a)^{\nu-1}}{(\nu-1)!}(\log x)^n a^n +$$

$$+ \int_{a \log x}^{\log x} \log\left(1 - \frac{w}{\log x}\right) \frac{(\log w)^{\nu-2}}{(\nu-2)!} w^{n-1}\, dw$$

$$+ n \int_{a \log x}^{\log x} \log\left(1 - \frac{w}{\log x}\right) \frac{(\log w)^{\nu-1}}{(\nu-1)!} w^{n-1}\, dw +$$

$$+ \theta \left\{ b_{105} \frac{(l_2 x)^{\nu+2}}{(\nu+2)!} (\log x)^{n-1} \nu^3 \right\}.$$

This is true for $\nu \geqslant 2$. When $\nu = 1$ the first integral is absent. Now

$$\int_{a \log x}^{\log x} \log\left(1 - \frac{w}{\log x}\right) \frac{(\log w)^{\nu-2}}{(\nu-2)!} w^{n-1}\, dw$$

$$= (\log x)^n \int_a^1 \log(1-z) \frac{(\overline{\log z \log x})^{\nu-2}}{(\nu-2)!} z^{n-1}\, dz$$

$$= -(\log x)^n \sum_{\mu=1}^{\infty} \frac{1}{\mu} \int_a^1 \frac{(\log z \log x)^{\nu-2}}{(\nu-2)!} z^{n+\mu-1}\, dz$$

$$= -(\log x)^n \left[\sum_{t'=1}^{\nu-1} (-1)^{t'-1} \frac{(\overline{\log z \log x})^{\nu-t'-1}}{(\nu-t'-1)!} \sum_{\mu=1}^{\infty} \frac{z^{n+\mu}}{\mu(n+\mu)^{t'}} \right]_{z=a}^{z=1} \tag{193}$$

by virtue of (125′). Putting $\nu+1$ for ν in (193), we get on multiplying by n,

$$n\int_{a\log x}^{\log x}\log\left(1-\frac{w}{\log x}\right)\frac{(\log w)^{\nu-1}}{(\nu-1)!}w^{n-1}\,dw$$

$$=-(\log x)^n\left[\sum_{t'=1}^{\nu}(-1)^{t'-1}\frac{(\log z\overline{\log x})^{\nu-t'}}{(\nu-t')!}\sum_{\mu=1}^{\infty}\frac{n\,z^{n+\mu}}{\mu(n+\mu)^{t'}}\right]_{z=a}^{z=1}$$

$$=-(\log x)^n\left[\frac{(\log z\overline{\log x})^{\nu-1}}{(\nu-1)}\sum_{\mu=1}^{\infty}\frac{n\,z^{n+\mu}}{\mu\,(n+\mu)}\right]_{z=a}^{z=1}-$$

$$-(\log x)^n\left[\sum_{t'=1}^{\nu-1}(-1)^{t'}\frac{(\log z\overline{\log x})^{\nu-t'-1}}{(\nu-t'-1)!}\sum_{\mu=1}^{\infty}\frac{n\,z^{n+\mu}}{\mu\,(n+\mu)^{t'+1}}\right]_{z=a}^{z=1}. \tag{194}$$

From (192)-(194) the lemma follows on taking $b_{84}\geqslant b_{105}$ and noting that

$$\frac{1}{\mu(n+\mu)^{t'}}-\frac{n}{\mu(n+\mu)^{t'+1}}=\frac{1}{(n+\mu)^{t'+1}}.$$

b_{84} has to satisfy one more condition, namely the one stated in the remark just below equation (182). This is evidently possible.

LEMMA 30. *If* $2\leqslant\nu\leqslant l_2x$; $1\leqslant n\leqslant 10\,l_2x$, *then*

$$\sum_{r=0}^{\nu-2}B_r\sum_{t=1}^{\nu-r-1}\frac{(-1)^{t-1}}{n^t}\sum_{p\leqslant x^{1-a}}\frac{(\log x-\log p)^n}{p}\frac{(l_2\overline{x/p})^{\nu-r-t-1}}{(\nu-r-t-1)!}$$

$$=\sum_{r=0}^{\nu-2}B_r\sum_{t=1}^{\nu-r-1}\frac{(-1)^{t-1}}{n^t}T(\nu-r-t-1)$$

$$=(\log x)^n\sum_{r=0}^{\nu-2}B_r\sum_{t=1}^{\nu-r-1}(-1)^t\frac{(l_2x)^{\nu-r-t-1}}{(\nu-r-t-1)!}\sum_{\mu=1}^{\infty}\frac{1}{\mu\,(n+\mu)^t}+$$

$$+(\log x)^n\sum_{r=0}^{\nu-2}B_r\sum_{t=1}^{\nu-r-1}(-1)^{t-1}\frac{(l_2x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!}\sum_{\mu=1}^{\infty}\frac{1}{\mu\,(n+\mu)^t}+$$

$$+\log(1-a)(\log x)^n a^n\sum_{r=0}^{\nu-2}B_r\sum_{t=1}^{\nu-r-1}(-1)^{t-1}\frac{(l_2x^a)^{\nu-r-t-1}}{(\nu-r-t-1)!}\frac{1}{n^t}+$$

$$+ (\log x)^n (l_2 x + B) \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \frac{1}{n^t} +$$

$$+ \theta \left\{ b_{106} (\log x)^{n-1} g(\nu-1, x) \frac{l_2 x}{\nu} \right\}. \tag{195}$$

PROOF. We can use Lemma 29 if $1 < \nu - r - t - 1 + 1 < l_2 x$. This is true since $l_2 x > \nu > 2$ and $t < \nu - r - 1$, $r \geqslant 0$, $t \geqslant 1$. The dominant term on the right side of (195) follows after easy calculations as in (130) and (131). Also the modulus of the error term is not greater than

$$b_{84} (\log x)^{n-1} \sum_{r=0}^{\nu-2} |B_r| \sum_{t=1}^{\nu-r-1} \frac{(l_2 x)^{\nu-r-t+2}}{(\nu-r-t+2)!} \nu^3$$

$$\leqslant b_{107} (\log x)^{n-1} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \nu^4 \sum_{r=0}^{\infty} |B_r|, \qquad \text{since } \nu < l_2 x;$$

$$= b_{108} (\log x)^{n-1} \frac{(l_2 x)^{\nu+1}}{(\nu+1)!} \nu^4$$

$$\leqslant b_{109} (\log x)^{n-1} g(\nu-1, x) \frac{l_2 x}{\nu}, \qquad \text{by (102).}$$

Hence the lemma follows immediately on taking $b_{106} \geqslant b_{109}$.

LEMMA 31. *Under the conditions of* (103),

$$\sum_{l=0}^{n} \binom{n}{l} \sum_{p \leqslant x^{1-\alpha}} \frac{(\log p)^l}{p} S_{\nu-1}(x/p, n-l) +$$

$$+ \sum_{q_{\nu-1} \leqslant x^\alpha} \frac{(\log q_{\nu-1})^{n-l}}{q_{\nu-1}} S(x/q_{\nu-1}, l) - S(x^{1-\alpha}, l).\, S_{\nu-1_\alpha}(x^\alpha, n-l)$$

$$= (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^t \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{\mu=1}^{\infty} \frac{1}{\mu(n+\mu)^t} +$$

$$+ (\log x)^n (l_2 x + B) \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \frac{1}{n^t} +$$

$$+ (\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \times$$

$$\times \sum_{l=1}^{n-1} \sum_{s=1} \frac{(-1)^{s-1}}{s} \binom{l-1}{s-1} \binom{n}{l} \left\{ \frac{1}{(n-l)^t} - \frac{1}{(n-l+s)^t} \right\} +$$

$$+ (\log x)^n \sum_{r=0}^{\nu-1} B_r \sum_{t=0}^{\nu-r-1} (-1)^t \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{s=1}^{n} \frac{(-1)^{s-1}}{s} \binom{n-1}{s-1} \frac{1}{s^t} +$$

$$+ \theta \left\{ b (\log x)^{n-1} g(\nu-1, x)\, l_2 x \left(1 + \frac{b_{110}}{\nu}\right) \right\}. \tag{196}$$

PROOF. This follows immediately from Lemmas 26, 27, 28, and 30 by taking

$$b_{110} > b_{57} + b_{58} + b_{80} + b_{106}, \qquad \text{since } b \geqslant 1.$$

LEMMA 32. *If* $t \geqslant 1$,

$$- \sum_{\mu=1}^{\infty} \frac{1}{\mu (n+\mu)^t} + \sum_{l=1}^{n-1} \sum_{s=1}^{l} \frac{(-1)^{s-1}}{s} \binom{l-1}{s-1} \binom{n}{l} \times$$

$$\times \left\{ \frac{1}{(n-l)^t} \frac{1}{(n-l+s)^t} \right\} + \sum_{s=1}^{n} \frac{(-1)^s}{s} \binom{n-1}{s-1} \frac{1}{s^t}$$

$$= \sum_{t'=1}^{t-1} \frac{1}{n^{t-t'}} \zeta(t'+1) - \frac{t+1}{n^{t+1}},$$

the empty sum on the right side when $t = 1$ *being regarded as equal to* 0.

PROOF: For $t \geqslant 1$,

$$\sum_{l=1}^{n-1} \sum_{s=1}^{l} \frac{(-1)^{s-1}}{s} \binom{l-1}{s-1} \binom{n}{l} \frac{1}{(n-l)^t}$$

$$= \sum_{l=1}^{n-1} \frac{1}{l} \binom{n}{l} \frac{1}{(n-l)^t} \sum_{s=1}^{l} (-1)^{s-1} \binom{l}{s},$$

$$\text{since } \frac{1}{l} \binom{l}{s} = \frac{1}{s} \binom{l-1}{s-1},$$

$$= \sum_{l=1}^{n-1} \frac{1}{l} \binom{n}{l} \frac{1}{(n-l)^t} = \sum_{\lambda=1}^{n-2} \frac{1}{\lambda} \binom{n}{\lambda} \frac{1}{(n-\lambda)^t} + \frac{n}{n-1}. \tag{197}$$

Also, for $t > 1$,

$$\sum_{l=1}^{n-1}\sum_{s=1}^{l}\frac{(-1)^{s-1}}{s}\binom{l-1}{s-1}\binom{n}{l}\frac{1}{(n-l+s)^t}$$

$$=\frac{1}{n^t}\sum_{l=1}^{n-1}\frac{(-1)^{l-1}}{l}\binom{n}{l}+$$

$$+\sum_{l=1}^{n-1}\sum_{s=1}^{l-1}\frac{(-1)^{s-1}}{s}\binom{l-1}{s-1}\binom{n}{l}\frac{1}{(n-l+s)^t}$$

$$=\left\{\frac{1}{n^t}\sum_{\mu=1}^{n}\frac{1}{\mu}+\frac{(-1)^n}{n^{t+1}}\right\}+$$

$$+\sum_{\lambda=1}^{n-2}\frac{1}{(n-\lambda)^t}\sum_{l=\lambda+1}^{n-1}\frac{(-1)^{l-\lambda-1}}{l-\lambda}\binom{l-1}{l-\lambda-1}\binom{n}{l},$$

on using (146) for the first sum and putting $l-s=\lambda$ in the second sum; the last expression is

$$=\frac{1}{n^t}\sum_{\mu=1}^{n}\frac{1}{\mu}+\frac{(-1)^n}{n^{t+1}}+$$

$$+\sum_{\lambda=1}^{n-2}\frac{(-1)^{\lambda-1}}{\lambda(n-\lambda)^t}\sum_{l=\lambda+1}^{n-1}(-1)^l\binom{l-1}{\lambda-1}\binom{n}{l},$$

since

$$\binom{l-1}{l-\lambda-1}\frac{1}{l-\lambda}=\frac{1}{l}\binom{l}{l-\lambda}=\frac{1}{l}\binom{l}{\lambda}=\frac{1}{\lambda}\binom{l-1}{\lambda-1}\text{ for }\lambda \geqslant 1;$$

$$=\frac{1}{n^t}\sum_{\mu=1}^{n}\frac{1}{\mu}+\frac{(-1)^n}{n^{t+1}}+$$

$$+\sum_{\lambda=1}^{n-2}\frac{1}{\lambda(n-\lambda)^t}\left\{-1+\binom{n}{\lambda}+(-1)^{n+\lambda}\binom{n-1}{\lambda-1}\right\}, \tag{198}$$

by virtue of (145c). Now

$$\sum_{\lambda=1}^{n-2}\frac{(-1)^{n+\lambda}}{\lambda(n-\lambda)^t}\binom{n-1}{\lambda-1}=\sum_{\lambda=1}^{n-2}\frac{(-1)^{n+\lambda}}{n}\frac{\binom{n}{\lambda}}{(n-\lambda)^t}=\sum_{s=2}^{n-1}\frac{(-1)^s}{n}\frac{\binom{n}{s}}{s^t}$$

$$=\sum_{s=1}^{n}\frac{(-1)^s}{s}\frac{\binom{n-1}{s-1}}{s^t}+\frac{(-1)^{n-1}}{n^{t+1}}+1. \tag{199}$$

Hence from (197)-(199) we get, for $t > 1$,

$$\sum_{l=1}^{n-1}\sum_{s=1}^{l}\frac{(-1)^{s-1}}{s}\binom{l-1}{s-1}\binom{n}{l}\left\{\frac{1}{(n-l)^t}-\frac{1}{(n-l+s)^t}\right\}+$$

$$+\sum_{s=1}^{n}\frac{(-1)^s}{s}\frac{\binom{n-1}{s-1}}{s^t}.$$

$$=-\frac{1}{n^t}\sum_{\mu=1}^{n}\frac{1}{\mu}+\sum_{\lambda=1}^{n-2}\frac{1}{\lambda}\frac{1}{(n-\lambda)^t}+\frac{1}{n-1}$$

$$=-\frac{1}{n^t}\sum_{\lambda=1}^{n}\frac{1}{\lambda}+\sum_{\lambda=1}^{n-1}\frac{1}{\lambda(n-\lambda)^t}. \tag{200}$$

Now, for $t \geqslant 1$,

$$\sum_{\lambda=1}^{n-1}\frac{1}{\lambda(n-\lambda)^t}-\sum_{\lambda=1}^{n-1}\frac{1}{\lambda\, n^t}=\sum_{\lambda=1}^{n-1}\frac{n^t-(n-\lambda)^t}{n^t(n-\lambda)^t}\cdot\frac{1}{\lambda}$$

$$=\frac{1}{n^t}\sum_{m=1}^{n-1}\frac{1}{m}+\frac{1}{n^{t-1}}\sum_{m=1}^{n-1}\frac{1}{m^2}+\dots+\frac{1}{n}\sum_{m=1}^{n-1}\frac{1}{m^t},$$

$$=-\frac{t}{n^t}+\frac{1}{n^t}\sum_{m=1}^{n}\frac{1}{m}+\frac{1}{n^{t-1}}\sum_{m=1}^{n}\frac{1}{m^2}+\dots+\frac{1}{n}\sum_{m=1}^{n}\frac{1}{m^t}.$$

Hence, for $t \geqslant 1$,

$$\sum_{\lambda=1}^{n-1}\frac{1}{\lambda(n-\lambda)^t}-\frac{1}{n^t}\sum_{\lambda=1}^{n}\frac{1}{\lambda}=-\frac{t+1}{n^t}+\frac{1}{n^t}\sum_{m=1}^{n}\frac{1}{m}+$$

$$+\frac{1}{n^{t-1}}\sum_{m=1}^{n}\frac{1}{m^2}+\dots+\frac{1}{n}\sum_{m=1}^{n}\frac{1}{m^t}. \tag{201}$$

Also, for $t \geqslant 2$,

$$\frac{1}{n^t}\cdot\frac{1}{\mu}-\frac{1}{\mu(n+\mu)^t}=\frac{1}{n^t(n+\mu)}+\frac{1}{n^{t-1}(n+\mu)^2}+\dots+\frac{1}{n(n+\mu)^t}$$

or

$$-\frac{1}{\mu(n+\mu)^t}=-\frac{1}{n^t}\cdot\frac{n}{\mu(n+\mu)}+\frac{1}{n^{t-1}(n+\mu)^2}+\dots+\frac{1}{n}\cdot\frac{1}{(n+\mu)^t}.$$

Hence

$$\left.\begin{aligned} -\sum_{\mu=1}^{\infty}\frac{1}{\mu(n+\mu)^t} &= -\frac{1}{n^t}\sum_{\mu=1}^{\infty}\frac{n}{\mu(n+\mu)}+\frac{1}{n^{t-1}}\sum_{m=n+1}^{\infty}\frac{1}{m^2}+\\ &\qquad +\dots+\frac{1}{n}\sum_{m=n+1}^{\infty}\frac{1}{m^t}, \text{ if } t\geqslant 2,\\ -\sum_{\mu=1}^{\infty}\frac{1}{\mu(n+\mu)} &= -\frac{1}{n}\sum_{\mu=1}^{\infty}\frac{n}{\mu(n+\mu)}. \end{aligned}\right\} \quad (202)$$

Further

$$1+\frac{1}{2}+\dots+\frac{1}{n}=\int_0^1\frac{1-(1-y)^n}{y}\,dy=\int_0^1\frac{1-u^n}{1-u}\,du$$

$$=-\int_0^1 n\,u^{n-1}\log(1-u)\,du.$$

Hence

$$1+\frac{1}{2}+\dots+\frac{1}{n}=\sum_{\mu=1}^{\infty}\frac{n}{\mu(n+\mu)}. \quad (203)$$

From (200) — (203) the lemma follows.

LEMMA 33. *Under the conditions of* (103),

$$\begin{aligned} &\nu\, S_\nu(x, n)\\ &=(\log x)^n\sum_{r=0}^{\nu-2}B_r\sum_{t=1}^{\nu-r-1}(-1)^{t-1}\frac{(l_2\,x)^{\nu-r-t-1}}{(\nu-r-t-1)\,!}\left(-\frac{t+1}{n^{t+1}}\right)+\\ &\quad+(\log x)^n\sum_{r=0}^{\nu-3}B_r\sum_{t=2}^{\nu-r-1}(-1)^{t-1}\frac{(l_2\,x)^{\nu-r-t-1}}{(\nu-r-t-1)\,!}\sum_{t'=1}^{t-1}\frac{1}{n^{t-t'}}\zeta(t'+1)+\\ &\quad+(\log x)^n\,(l_2\,x+B)\sum_{r=0}^{\nu-2}B_r\sum_{t=1}^{\nu-r-1}(-1)^{t-1}\frac{(l_2\,x)^{\nu-r-t-1}}{(\nu-r-t-1)\,!}\,\frac{1}{n^t}+\\ &\quad+\frac{(\log x)^n}{n}\sum_{r=0}^{\nu-1}B_r\frac{(l_2\,x)^{\nu-r-1}}{(\nu-r-1)\,!}+\\ &\quad+\sum_{t=2}^{\nu-1}(-1)^{t-1}\sum_{p^t\leqslant x}\frac{1}{p^t}S_{\nu-t}(x/p^t, n)+\\ &\quad+\theta\left\{b\,(\log x)^{n-1}\,g\,(\nu-1, x)\,l_2\,x\left(1+\frac{b_{111}}{\nu}\right)\right\}. \end{aligned}$$

PROOF. Follows immediately from Lemmas 18, 31 and 32 and taking $b_{111} > b_{110} + b_2$. It may be noted that

$$\sum_{s=1}^{n} \frac{(-1)^{s-1}}{s} \binom{n-1}{s-1} = \frac{1}{n} \sum_{s=1}^{n} (-1)^s \binom{n}{s} = \frac{1}{n}.$$

LEMMA 34. *If* $l_2 x \geqslant \nu \geqslant 3$; $1 \leqslant n < 10(l_2 x - \nu \log 2)$, *and* (88) *is true for the indices* 1, 2, ..., $(\nu - 2)$, *then*

$$\sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p^t \leqslant x} \frac{1}{p^t} S_{\nu-t}(x/p^t, n)$$
$$= (\log x)^n \sum_{t=2}^{\nu-1} (-1)^{t-1} \lambda(t) \sum_{r=0}^{\nu-t-1} B_r \times$$
$$\times \sum_{t'=1}^{\nu-r-t} (-1)^{t'-1} \frac{(l_2 x)^{\nu-r-t-t'}}{(\nu-r-t-t')!} \frac{1}{n^{t'}} +$$
$$+ \theta \left\{ b_{112} (\log x)^{n-1} g(\nu - 1, x) \right\}.$$

PROOF.

$$\sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p^t \leqslant x} \frac{1}{p^t} S_{\nu-t}(x/p^t, n)$$
$$= \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p \leqslant (\log x)} \frac{1}{p^t} S_{\nu-t}(x/p^t, n) +$$
$$+ \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{x^{1/t} \geqslant p > \log x} \frac{1}{p^t} S_{\nu-t}(x/p^t, n)$$
$$= \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p \leqslant \log x} \frac{(\log \overline{x/p^t})^n}{p^t} \sum_{r=0}^{\nu-t-1} B_r \times$$
$$\times \sum_{t'=1}^{\nu-r-t} (-1)^{t'-1} \frac{(l_2 \overline{x/p^t})^{\nu-r-t-t'}}{(\nu-r-t-t')!} \frac{1}{n^{t'}} +$$
$$+ \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{p \leqslant \log x} \frac{1}{p^t} E_{\nu-t}(x/p^t, n) +$$
$$+ \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{x^{1/t} \geqslant p > \log x} \frac{1}{p^t} S_{\nu-t}(x/p^t, n), \qquad \text{by virtue of (84)},$$
$$= \Sigma_1 + \Sigma_2 + \Sigma_3, \text{ say.} \tag{204}$$

It may be explicitly mentioned that $(\log x)^t < (\log x)^{l_2 x} < \sqrt{x}$, if $l x > 3$.

Now, for $2 < t < \nu - 1$,

$$\nu - t < \log\log\left\{\frac{x}{(\log x)^t}\right\}, \text{ if } l_2 x > \nu > 3. \qquad (204')$$

For, to prove (204′) it is sufficient to show that

$$\nu > -\log\left(1 - \frac{t\, l_2 x}{\log x}\right), \text{or } t > t\,\frac{l_2 x}{\log x}\cdot\frac{1}{1 - t\, l_2 x/\log x},$$

and this is true if $1 > l_2 x/\log x - (l_2 x)^2$, when $l_2 x > 3$. Now, if $2 \leqslant t \leqslant \nu - 1$ we have by our assumption,

$$|E_{\nu-t}(x, n)| < b(\log x)^{n-1} g(\nu - t, x),$$

if $l_2 x > \nu - t > 1$ and $1 < n < 10\,(l_2 x - \overline{\nu - t} \log 2)$. Hence by virtue of (204′) we see that if

$$l_2 x > \nu > 3;\ 1 \leqslant n \leqslant 10\,(l_2 x - \nu \log 2);\ p \leqslant \log x,$$

then

$$\begin{aligned} |E_{\nu-t}(x/p^t, n)| &\leqslant b(\log \overline{x/p^t})^{n-1} g(\nu - t, \overline{x/p^t}) \\ &\leqslant b(\log)^{n-1} g(\nu - t, x) \\ &\leqslant b(\log x)^{n-1} g(\nu - 1, x). \end{aligned}$$

Hence for $l_2 x > \nu > 3$, $1 \leqslant n < 10\,(l_2 x - \nu \log 2)$,

$$\begin{aligned} \left|\sum\nolimits_2\right| &\leqslant b(\log x)^{n-1} g(\nu - 1, x) \sum_{t=2}^{\nu-1} \sum_{p \leqslant \log x} \frac{1}{p^t} \\ &\leqslant b(\log x)^{n-1} g(\nu - 1, x) \sum_{t=2}^{\infty} \sum_{p} \frac{1}{p^t} \\ &= b_{113}(\log x)^{n-1} g(\nu - 1, x). \end{aligned} \qquad (205)$$

$$\begin{aligned} \left|\sum\nolimits_3\right| &\leqslant \sum_{t=2}^{\nu-1} \sum_{x/2 \geqslant p^t > (\log x)^t} \frac{1}{p^t} S_{\nu-t}(x/p^t, n) \\ &\leqslant b_{114} \frac{(l_2 x)^\nu}{\nu!} (\log x)^n \nu \sum_{t=2}^{\nu-1} \sum_{x/2 \geqslant p^t > (\log x)^t} 1/p^t, \end{aligned}$$

by Lemma 15, on noting that $\nu \leqslant l_2 x$;

$$< b_{114} \frac{(l_2 x)^\nu}{\nu\,!} (\log x)^n \nu \sum_{p > \log x} \sum_{t=2}^{\infty} 1/p^t$$

$$\leqslant 2\, b_{114} \frac{(l_2 x)^\nu}{\nu\,!} (\log x)^n \nu \sum_{p > \log x} 1/p^2$$

$$< b_{115} \frac{(l_2 x)^\nu}{\nu\,!} (\log x)^{n-1} \nu$$

$$\leqslant b_{116} (\log x)^{n-1} g(\nu - 1, x), \qquad \text{by (102)}. \tag{206}$$

Also, as in (182A), for $p \leqslant \log x$; $t < \nu \leqslant l_2 x$; $1 < n \leqslant 10\, l_2 x$,

$$(\log \overline{x/p^t})^n (l_2 \overline{x/p^t})^{\nu-r-t-t'}$$
$$= (\log x)^n (l_2 x)^{\nu-r-t-t'} \left\{ 1 + \theta \left(b_{117} \frac{\nu (l_2 x)^2}{\log x} \right) \right\}.$$

Hence

$$\sum\nolimits_1 = (\log x)^n \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{r=0}^{\nu-t-1} B_r \times$$
$$\times \sum_{t'=1}^{\nu-r-t} \frac{(-1)^{t'-1}}{n^{t'}} \frac{(l_2 x)^{\nu-r-t-t'}}{(\nu - r - t - t')\,!} \sum_{p \leqslant lx} 1/p^t +$$
$$+ \theta \left\{ b_{117} (\log x)^n \frac{\nu (l_2 x)^2}{\log x} \frac{(l_2 x)^{\nu-3}}{(\nu-3)\,!} \nu \times \right.$$
$$\left. \times \left(\sum_{r=0}^{\infty} | B_r | \sum_{t=2}^{\infty} \sum_{p \leqslant lx} 1/p^t \right) \right\}$$

$$= (\log x)^n \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{r=0}^{\nu-t-1} B_r \sum_{t'=1}^{\nu-r-t} \frac{(-1)^{t'-1}}{n^{t'}} \times$$
$$\times \frac{(l_2 x)^{\nu-r-t-t'}}{(\nu - r - t - t')\,!} \lambda(t) -$$

$$- (\log x)^n \sum_{t=2}^{\nu-1} (-1)^{t-1} \sum_{r=0}^{\nu-t} B_r \sum_{t'=1}^{\nu-r-t} \frac{(-1)^{t'-1}}{n^{t'}} \times$$
$$\times \frac{(l_2 x)^{\nu-r-t'-t}}{(\nu - r - t - t')\,!} \times \sum_{p > lx} 1/p^t +$$

$$+ \theta \left\{ b_{118} (\log x)^{n-1} \frac{(l_2 x)^\nu}{\nu\,!} \nu^4 \right\}.$$

Since

$$\sum_{t=2}^{\infty} \sum_{p \leqslant lx} 1/p^t < 2 \sum_{p} 1/p^2$$

$$= (\log x)^n \sum_{t=2}^{\nu-1} (-1)^{t-1} \lambda(t) \sum_{r=0}^{\nu-t-1} B_r \times$$

$$\times \sum_{t'=1}^{\nu-r-t} \frac{(-1)^{t'-1}}{n^{t'}} \frac{(l_2 x)^{\nu-r-t-t'}}{(\nu - r - t - t')\,!} +$$

$$+ \theta \left\{ (\log x)^n \frac{(l_2 x)^{\nu-3}}{(\nu-3)\,!} \nu \left(\sum_{r=0}^{\infty} |B_r| \right) \sum_{t=2}^{\infty} \sum_{p > lx} 1/p^t \right\} +$$

$$+ \theta \left\{ b_{118} (\log x)^{n-1} \frac{(l_2 x)^{\nu}}{\nu\,!} \nu^4 \right\}$$

$$= (\log x)^n \sum_{t=2}^{\nu-1} (-1)^{t-1} \lambda(t) \sum_{r=0}^{\nu-t-1} B_r \times$$

$$\times \sum_{t'=1}^{\nu-r-t} \frac{(-1)^{t'-1}}{n^{t'}} \frac{(l_2 x)^{\nu-r-t-t'}}{(\nu - r - t - t')\,!} +$$

$$+ \theta \left\{ b_{119} (\log x)^{n-1} \frac{(l_2 x)^{\nu}}{\nu\,!} \nu^4 \right\}, \tag{207}$$

since

$$\sum_{t=2}^{\infty} \sum_{p > lx} 1/p^t < 2 \sum_{p > lx} 1/p^2 < \frac{2}{\log x}. \tag{208}$$

The result follows immediately from (204), (205)-(207), since by virtue of (102),

$$b_{119} (\log x)^{n-1} \frac{(l_2 x)^{\nu}}{\nu\,!} \nu^4 \leqslant b_{120} (\log x)^{n-1} g(\nu - 1, x).$$

LEMMA 35. *If* (88) *is true for the indices* 1, 2, ... $(\nu - 1)$ *and* $2 < \nu \leqslant \nu'$, $1 \leqslant n \leqslant 10\,(l_2 x - \nu \log 2)$, *then when* $l_2 x \geqslant \nu$,

$\nu S_\nu (x, n)$

$$= -(\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu - r - t - 1)\,!} \frac{t+1}{n^{t+1}} +$$

$$+(\log x)^n \sum_{r=0}^{\nu-3} B_r \sum_{t=2}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{t'=1}^{t-1} \frac{1}{n^{t-t'}} \zeta(t'+1) +$$

$$+(\log x)^n (l_2 x + B) \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \frac{1}{n^t} +$$

$$+\frac{(\log x)^n}{n} \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x)^{\nu-r-1}}{(\nu-r-1)!} +$$

$$+(\log x)^n \sum_{t=2}^{\nu-1} (-1)^{t-1} \lambda(t) \sum_{r=0}^{\nu-t-1} B_r \times$$

$$\times \sum_{t'=1}^{\nu-r-t} (-1)^{t'-1} \frac{(l_2 x)^{\nu-r-t-t'}}{(\nu-r-t-t')!} \frac{1}{n^{t'}} +$$

$$+\theta \{ b(\log x)^{n-1} g(\nu, x)\, \nu \},$$

empty sums when $\nu = 2$ *being regarded as equal to* 0.

PROOF. The dominant expression on the right side follows from Lemmas 33 and 34. Also, the modulus of the error term does not exceed, by virtue of Lemmas 33 and 34,

$$b\,(\log x)^{n-1} g\,(\nu-1, x)\, l_2\, x \left(1 + \frac{b_{111}}{\nu} + \frac{b_{112}}{\nu}\right), \quad \text{since } b \geqslant 1,$$

$$= b\,(\log x)^{n-1} g\,(\nu-1, x) \frac{l_2 x}{(\nu+1)} \nu \left(1 + \frac{1}{\nu}\right)\left(1 + \frac{b_{121}}{\nu}\right)$$

$$= b\,(\log x)^{n-1} g\,(\nu-1, x) \frac{l_2 x}{(\nu+1)} \nu \left(1 + \frac{b_{122}}{\nu}\right)$$

$$= b\,(\log x)^{n-1} g\,(\nu-1, x) \frac{l_2 x}{(\nu+1)} \left(1 + \frac{c'}{\nu}\right) \nu, \tag{212}$$

by virtue of (16). Also

$$g\,(\nu-1, x) \frac{(l_2 x)^{\nu+1}}{\nu+1} h\,(\nu-1)\,(1 + c'/\nu)$$

$$= (\nu-1)^{1/2} \frac{(l_2 x)^{\nu+1}}{(\nu+1)\,!} h\,(\nu-1)\,(1 + c'/\nu)$$

$$\leqslant (\nu)^{1/2} \frac{(l_2 x)^{\nu+1}}{(\nu+1)\,!} h\,(\nu) = g\,(\nu, x).$$

Hence the result follows from (212).

LEMMA 36.

$$\sum_{r=0}^{\nu-3} B_r \sum_{t=2}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \sum_{t'=1}^{t-1} \frac{1}{n^{t'}} \zeta(t'+1)$$

$$= \sum_{t=2}^{\nu-1} (-1)^{t-1} \zeta(t) \sum_{r=0}^{\nu-t-1} B_r \sum_{t'=1}^{\nu-r-t} (-1)^{t'-1} \frac{(l_2 x)^{\nu-r-t-t'}}{(\nu-r-t-t')!} \frac{1}{n^{t'}}.$$

PROOF. Coefficient of $\zeta(\mu)$ on the left is equal to

$$\sum_{r=0}^{\nu-\mu-1} B_r \sum_{t=\mu}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \frac{1}{n^{t-\mu+1}}$$

$$= (-1)^{\mu-1} \sum_{r=0}^{\nu-\mu-1} B_r \sum_{t'=1}^{\nu-r-\mu} (-1)^{t'-1} \frac{(l_2 x)^{\nu-r-\mu-t'}}{(\nu-r-t-t')!} \frac{1}{n^{t'}},$$

on putting $t-\mu+1=t'$. The lemma follows immediately from this on writing t for μ.

COROLLARY. *It follows from this and from Lemma 35 that if* (88) *is true for the indicies* 1, 2, , ... $(\nu-1)$; $2 \leqslant \nu \leqslant \nu'$, $1 \leqslant n \leqslant 10\,(l_2 x - \nu \log 2)$, *then when* $l_2 x \geqslant \nu$,

$$\nu S_\nu(x, n)$$

$$= -(\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \frac{(t+1)}{n^{t+1}} +$$

$$+ \frac{(\log x)^n}{n} \sum_{r=0}^{\nu-1} B_r \frac{(l_2 x)^{\nu-r-1}}{(\nu-r-1)!} +$$

$$+ (l_2 x + B)(\log x)^n \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \frac{1}{n^t} +$$

$$+ (\log x)^n \sum_{t=2}^{\nu-1} (-1)^{t-1} A(t) \sum_{r=0}^{\nu-t-1} B_r \times$$

$$\times \sum_{t'=1}^{\nu-r-1} (-1)^{t'-1} \frac{(l_2 x)^{\nu-r-t-t'}}{(\nu-r-t-t')!} \frac{1}{n^{t'}} +$$

$$+ \theta \left\{ b (\log x)^{n-1} g(\nu, x)\, \nu \right\}. \qquad (213)$$

LEMMA 37.

$$\sum_{r=1}^{\nu-1} r B_r \sum_{t=1}^{\nu-r} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} \frac{1}{n^t}$$

$$= B \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \frac{1}{n^t} +$$

$$+ \sum_{t=2}^{\nu-1} (-1)^{t-1} A(t) \sum_{r=0}^{\nu-t-1} B_r \sum_{t'=1}^{\nu-r-t} (-1)^{t'-1} \times$$

$$\times \frac{(l_2 x)^{\nu-r-t-t'}}{(\nu-r-t-t')!} \frac{1}{n^{t'}}.$$

PROOF.

$$\sum_{r=1}^{\nu-1} r B_r \sum_{t=1}^{\nu-r} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} \frac{1}{n^t}$$

$$= \sum_{t=1}^{\nu-1} \sum_{r=1}^{\nu-t} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} \frac{1}{n^t} r B_r$$

$$= \sum_{t=1}^{\nu-1} \sum_{r=1}^{\nu-t} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} \frac{1}{n^t} \{B_{r-1} B - B_{r-2} A(2) + \dots\},$$

by virtue of (13).

The coefficient of $A(\mu)$ on the right side is equal to

$$(-1)^{\mu-1} \sum_{t=1}^{\nu-\mu} \sum_{r=\mu}^{\nu-t} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} \frac{1}{n^t} \cdot \quad B_{r-\mu}$$

$$= (-1)^{\mu-1} \sum_{t=1}^{\nu-\mu} \sum_{r=0}^{\nu-t-\mu} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-\mu}}{(\nu-r-t-\mu)!} \frac{1}{n^t} B_r$$

$$= (-1)^{\mu-1} \sum_{r=0}^{\nu-\mu-1} B_r \sum_{t=1}^{\nu-r-\mu} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-\mu}}{(\nu-r-t-\mu)!} \frac{1}{n^t}.$$

From this the result follows immediately.

LEMMA 38. *If* (88) *is true for the indices* 1, 2, ... , $(\nu - 1)$ *then it is also true for the index* ν *provided* $2 \leqslant \nu \leqslant \nu'$.

PROOF. For $\nu = 2$ the result is directly verified by (213). Hence we may suppose $\nu' \geqslant \nu \geqslant 3$. Now

$$l_2 x \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \frac{1}{n^t}$$

$$= \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} \frac{(\nu-r-t)}{n^t}$$

$$= \nu \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} \frac{1}{n^t} -$$

$$- \sum_{r=0}^{\nu-2} r B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} \frac{1}{n^t} -$$

$$- \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} \frac{1}{n^t}$$

$$= \nu \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} \frac{1}{n^t} -$$

$$- \sum_{r=1}^{\nu-2} r B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-1}}{(\nu-r-t)!} \frac{1}{n^t} -$$

$$- \sum_{r=0}^{\nu-2} B_r \frac{(l_2 x)^{\nu-r-1}}{(\nu-r-t)!} \frac{1}{n} +$$

$$+ \sum_{r=0}^{\nu-3} B_r \sum_{t=1}^{\nu-r-2} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t-1}}{(\nu-r-t-1)!} \frac{(t+1)}{n^{t+1}}. \tag{214}$$

Hence we get from (213), Lemma 37, and (214):

If $2 \leqslant \nu \leqslant \nu'$. (88) is true for the indices $1, 2, \ldots, (\nu-1)$; $1 \leqslant n \leqslant 10\,(l_2 x - \nu \log 2)$ then when $l_2 x \geqslant \nu$,

$$\nu S_\nu(x, n)$$

$$= (\log x)^n \left\{ \nu \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu-r-t)!} \frac{1}{n^t} \right\} +$$

$$+ (\log x)^n \sum_{r=1}^{\nu-1} r B_r \frac{(-1)^{\nu-r-1}}{n^{\nu-r}} + (\log x)^n \frac{B_{\nu-1}}{n} -$$

$$- (\log x)^n \sum_{r=0}^{\nu-2} B_r (-1)^{\nu-r-2} \frac{(\nu - r)}{n^{\nu-r}} +$$

$$+ \theta \left\{ b (\log x)^{n-1} g(\nu, x) \nu \right\}$$

$$= (\log x)^n \left\{ \nu \sum_{r=0}^{\nu-2} B_r \sum_{t=1}^{\nu-r-1} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu - r - t)!} \frac{1}{n^t} + \right.$$

$$\left. + \nu \sum_{r=0}^{\nu-1} B_r \frac{(-1)^{\nu-r-1}}{n^{\nu-r}} \right\} +$$

$$+ \theta \left\{ b (\log x)^{n-1} g(\nu, x) \nu \right\}$$

$$= \nu (\log x)^n \sum_{r=0}^{n-1} B_r \sum_{t=1}^{\nu-r} (-1)^{t-1} \frac{(l_2 x)^{\nu-r-t}}{(\nu - r - t)!} \frac{1}{n^t} +$$

$$+ \nu \theta \left\{ b (\log x)^{n-1} g(\nu, x) \right\}.$$

Dividing by ν we see that $| E_\nu (x, n) | \leqslant b(\log x)^{n-1} g(\nu, x)$, if $1 \leqslant n < 10 (l_2 - \nu \log 2)$ and $l_2 x \geqslant \nu$. Hence the result.

LEMMA 39. *If* $1 \leqslant \nu \leqslant \nu'$, $1 \leqslant n \leqslant 2 l_2 x$ *and* (23) *holds, then, when* $l_2 x \geqslant \nu$,

$$| E_\nu(x, n) | \leqslant b(\log x)^{n-1} g(\nu, x). \tag{215}$$

PROOF. By virtue of (91), (88) is true when $\nu = 1$. Also by Lemma 38, if (88) is true for the indices $1, 2, \ldots, (\nu - 1)$; $2 \leqslant \nu \leqslant \nu'$, then it is true for the index ν also. Hence (88) is true for $2 \leqslant \nu \leqslant \nu'$. From this (215) follows immediately, since $10 (\log x - \nu \log 2) = 10 (1 - \log 2) l_2 x \geqslant 2 l_2 x$ under the given conditions.

We shall close this section by proving one more result, namely

LEMMA 40. *If* $2 \leqslant \nu \leqslant \nu'$ *and* (23) *holds, then, when* $l_2 x \geqslant \nu + 1$,

$$\sum_{q_\nu \leqslant x^a} \frac{1}{q_\nu \{\log x - \log q_\nu\}} = \frac{1}{\log x} \sum_{r=0}^{\nu} B_r \frac{(l_2 x^a)^{\nu-r}}{(\nu - r)!} +$$

$$+ \frac{1}{\log x} \sum_{r=0}^{\nu-1} B_r \sum_{t=1}^{\nu-r} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t}}{(\nu - r - t)!} \sum_{n=1}^{\infty} \frac{a^n}{n^t} +$$

$$+ \theta \left\{ C_{19} \frac{g(\nu, x)}{(\log x)^2} \right\}. \tag{216}$$

PROOF : Let $m = [2\, l_2\, x - 2]$, then

$$\sum_{q_\nu \leqslant x^a} \frac{1}{q_\nu \{\log x - \log q_\nu\}}$$

$$= \frac{1}{\log x} \sum_{q_\nu \leqslant x^a} \frac{1}{q_\nu} + \sum_{n=1}^{m} \frac{1}{(\log x)^{n+1}} \sum_{q_{\nu-1} \leqslant x^a} \frac{(\log q_\nu)^n}{q_\nu} +$$

$$+ \theta \left\{ C_{20} \frac{1}{(\log x)^{m+2}} \sum_{q_{\nu-1} \leqslant x^a} \frac{(\log q_\nu)^{m+1}}{q_\nu} \right\}$$

$$= \frac{1}{\log x} S_\nu(x^a) + \sum_{n=1}^{m} \frac{1}{(\log x)^{n+1}} S_\nu (x^a, n) +$$

$$+ \theta \left\{ C_{20} \frac{1}{(\log x)^{m+2}} S_\nu (x^a, m+1) \right\}$$

$$= \frac{1}{\log x} \sum_{r=0}^{\nu} B_r \frac{(l_2\, x^a)^{\nu - r}}{(\nu - r)\,!} +$$

$$+ \frac{1}{\log x} \sum_{r=0}^{\nu-1} B_r \sum_{t=1}^{\nu-r} (-1)^{t-1} \frac{(l_2\, x^a)^{\nu-r-t}}{(\nu - r - t)\,!} \sum_{n=1}^{m} \frac{a^n}{n^t} +$$

$$+ \frac{1}{\log x} E_\nu (x^a) + \frac{1}{\log x} \sum_{n=1}^{m} \frac{E_\nu (x^a, n)}{(\log x)^n} +$$

$$+ \theta \left\{ C_{20} \frac{1}{(\log x)^{m+2}} S_\nu (x^a, m+1) \right\}, \text{ by (36) and (86).} \qquad (217)$$

Now by (38), if $l_2\, x > \nu + 1$,

$$| E_\nu (x^a) | < C_6 \frac{(l_2\, x)^\nu}{\nu\,!} \nu^{1/2} \frac{h(\nu)}{\log x} < C_6 \frac{g\,(\nu, x)}{\log x}. \qquad (218)$$

Also by Lemma 39 if $1 \leqslant n \leqslant 2\, l_2\, x^a$ and $l_2\, x^a > \nu \geqslant 1$,

$$| E_\nu (x^a, n) | < b\,(\log x)^{n-1}\, a^{n-1}\, g(\nu, x^a) < b\,(\log x)^{n-1}\, g(\nu, x)\, a^{n-1}.$$

Hence, if $m = [2\, l_2\, x - 2]$, $l_2\, x > \nu + 1$, we have

$$\sum_{n=1}^{m} \frac{E_\nu (x^a, n)}{(\log x)^n} < b\,(\log x)^{-1}\, g(\nu, x) \sum_{n=1}^{\infty} a^{n-1}$$

$$= 2\, b\, \frac{g(\nu, x)}{\log x}. \qquad (219)$$

Also by Lemma 15,

$$S_\nu(x^a, m+1) < b_7 \frac{(l_2 x^a + b_8)^\nu}{\nu\,!} \nu(\log x^a)^{m+1}$$
$$< C_{21} \frac{(l_2 x)^\nu}{\nu\,!} \nu(\log m)^{m+1} a^{m+1}$$
$$< 8\, C_{21} \frac{(l_2 x)^\nu}{\nu\,!} \nu(\log x)^{m+1} \frac{1}{2^{2l_2 x}}$$
$$< C_{22} (\log x)^m g(\nu, x), \qquad \text{by (102).} \qquad (220)$$

Also

$$\left|\sum_{r=0}^{\nu-1} B_r \sum_{t=1}^{\nu-r} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t}}{(\nu-r-t)\,!} \sum_{n=m+1}^{\infty} \frac{a^n}{n^t}\right| < 2a^{m+1} \frac{(l_2 x)^\nu}{\nu\,!} \nu \sum_{r=0}^{\infty} |B_r|$$
$$< C_{23} \frac{g(\nu, x)}{\log x}, \qquad \text{as in (220).} \qquad (221)$$

The result follows immediately from (217)-(221).

Corollary. *If* $2 \leqslant \nu \leqslant \nu'$, (23) *holds, then, when* $l_2 x > \nu + 1$,

$$\sum_{q_\nu \leqslant x^a} \pi\left(\frac{x}{q_\nu}\right) = \frac{x}{\log x} \sum_{r=0}^{\nu} B_r \frac{(l_2 x^a)^{\nu-r}}{(\nu-r)\,!} +$$
$$+ \frac{x}{\log x} \sum_{r=0}^{\nu-1} B_r \sum_{t=1}^{\nu-r} (-1)^{t-1} \frac{(l_2 x^a)^{\nu-r-t}}{(\nu-r-t)!} +$$
$$+ \sum_{n=1}^{\infty} \frac{a^n}{n^t} + \theta\left\{C_{24} \frac{x}{(\log x)^2} g(\nu, x)\right\}. \qquad (222)$$

Proof.

$$\left|\sum_{q_\nu \leqslant x^a} \pi\left(\frac{x}{q_\nu}\right) - \sum_{q_\nu \leqslant x^a} \frac{x}{q_\nu(\log x - \log q_\nu)}\right|$$
$$< C_{25}\, x \sum_{q_\nu \leqslant x^a} \frac{1}{q_\nu(\log x - \log q_\nu)^2} < C_{26} \frac{x}{(\log x)^2} \sum_{q_\nu \leqslant x^a} \frac{1}{q_\nu}$$
$$< C_{27} \frac{(l_2 x)^\nu}{\nu!} \nu \frac{x}{(\log x)^2}, \text{ by Lemma 15 and noting that } \nu \leqslant l_2 x,$$
$$\leqslant C_{28} \frac{x}{(\log x)^2} g(\nu, x), \qquad \text{by (102).}$$

Hence (222) follows immediately from the above lemma.

ON A CHARACTERIZATION OF MINIMALLY BICOMPACT SPACES: CORRECTIONS AND ADDITIONS

By K. PADMAVALLY

[Received April 10, 1953]

Theorem 1 of [2] states that *the condition for every closed set A and any limit point p of A, there exists a subset Z of A not containing p but of which p is the sole complete limit point is necessary and sufficient for the space to be minimally bicompact.* The proof ([2] page 2) of the sufficiency of this condition assumes that the closure of the set S under the new topology is $A-p$ where A is its closure under the given topology and p some limit point of S (not belonging to S). This is clearly not the most general method of weakening the given topology. Also the statement that this condition is sufficient for the space to be minimally bicompact is wrong, as shown in §1 below by an example of a T_1-space which satisfies the condition of Theorem 1 of [2], but is not minimally bicompact. It is to be noted that the error pointed out above does not affect the proof of the other part of the theorem, namely that the condition is necessary for the space to be minimally bicompact.

In line 10, page 1 of [2], 'bicompact' must be replaced by 'Hausdorff bicompact', for Alexandroff [1] considers Hausdorff only. The result proved by Alexandroff does not hold in general for all bicompact spaces or even for all T_1-bicompact spaces, as shown by a counter example (of a T_1-bicompact space possessing a point which has no convergence character) in § 2. It is however true that in any bicompact space at least one point has convergence character [cf. Theorem 1 of this note]. Also every bicompact space has a weaker topology under which it is bicompact and every point has convergence character [cf. Theorem 2 of this note].

§1. The following is an example of a bicompact T_1-space satisfying the condition of Theorem 1 of [2], which is not minimally bicompact.

Let R be a dense-in-itself minimally bicompact space and S any infinite set. Let R_1 be the space whose elements are all elements of R and S, in which every point belonging to S is its own neighbourhood and the neighbourhood system of any point p of the subset R consists of all sets of the form $V \cup (S\text{-}F)$ where V is any neighbourhood of p under the given topology of the space R and F any finite subset of S. The postulates for a neighbourhood system of a T_1-space are clearly satisfied. Also it is evident that the given space R is a subspace of R_1 and that S is an isolated set. It follows that the limit points (in the space R_1) of any set X are contained in R, and hence for any subset X of R_1 which is contained in R, the sets of limit points in the space R_1 and in the subspace R respectively are identical The space R_1 satisfies the conditions of Theorem 1 of [2], as shown by the following steps.

Consider any closed subset X of R_1. If $X \cap S$ is finite, then every limit point p of X in R_1 is a limit point of the subset $X \cap R$ Also $X \cap R$ is closed in the space R. Therefore since R is minimally bicompact, if follows from the latter part of Theorem 1 of [2] (which is still true), that there exists a subset of $X \cap R$ not containing p but of which p is the sole complete limit point in R and hence in R_1 also. If $X \cap S$ is infinite, $\overline{X \cap S} \supset R$. Therefore $X = \overline{X}$ also contains R, and every limit point p of X in R_1, is a point of R, which is a limit point of R in R, (R being dense-in-itself). Hence by the latter part of Theorem 1, R has a subset Z not containing p but of which p is the sole complete limit point in R and hence in R_1 also. Z is contained in X. Hence in all cases (for every limit point p of X) X has a subset Z not containing p but of which p is the sole complete limit point.

In order to prove that R_1 is bicompact, but not minimally bicompact, it is clearly sufficient to show that there exists a bicompact

topology on R_1 weaker than the topology defined above, and this can be done as follows.

Consider the topology of R_1, in which 1) every point of S is its own neighbourhood; 2) the neighbourhood system of some fixed point a of the subset R is the same as defined above, i.e. consists of all sets of the form $V \cup (S\text{-}F)$ where V is any neighbourhood of a in the space R, and F any finite subset of S; and 3) the neighbourhood system of any point p belonging to R different from a is the same as in the space R. The postulates for a neighbourhood system are clearly satisfied. Also this topology is weaker than the topology defined above. The space R_1 is bicompact under the new topology, since every infinite set X has a complete limit point (If $X \cap R$ has the same power as X, this latter set has a complete limit point p in R, R being bicompact, and p will be a complete limit point of $X \cap R$ and hence of X in R_1 also. If $X \cap R$ has power less than that of X, since X is an infinite set, $X \cap S$ has the same power as X. Then a is clearly a complete limit point of $X \cap S$ and hence of X in R_1).

It follows that the property of a bicompact space, that for every limit point p of a closed set A there exists a subset Z of A not containing p but of which p is the sole complete limit point, is not sufficient for the space to be minimally bicompact.

Consequently the first part of the result stated in [2], page 2, namely that the space R is minimally bicompact at a point p if every closed set A of which p is a limit point has a subset Z not containing p but of which p is the sole complete limit point, is also wrong.

§2. The result of Alexandroff [1] that in a Hausdorff bicompact space every point has convergence character, cannot be extended to all T_1-bicompact spaces. The following is an example of a T_1-bicompact space in which a point has no convergence character.

For every integer $n > 1$ let A_n denote the set of all points of the x-y plane of the form $\{(1 - 1/n - 1/m), 0\}$ for positive integral values of m. Let R denote the enumerable set whose elements consist of all elements of the sets $[A_n]_{n>1}$; their limit points, namely all points of the form $\{(1 - 1/n), 0\}$, $(n > 1)$, and the points $(1, 0)$, $(1, 1)$ on the x-y plane. Consider the topology on R in which the neighbourhood system of all points of R other than the point $(1, 1)$ are the intersections with R of their respective neighbourhood systems in the natural topology of the real number space, and the neighbourhoods of $(1,1)$ are all sets of the form $\sum_{n=n_0}^{\infty} A_n - \sum_{n=n_0}^{\infty} P_n$, where n_0 is an integer > 1, and P_n for each n is any finite subset of A_n. It can be easily seen that R is bicompact under this topology (since every infinite set has a complete limit point). Also no set flows towards the point $(1,1)$, since if the set X contains an infinite subset Y belonging to any of the sets A_n, the point $(1,1)$ has a neighbourhood disjoint with the set A_n, and hence with Y also, while if $X \cap A_n$ is finite for every A_n, $R - X$ will be a neighbourhood of $(1,1)$. Hence the point $(1,1)$ has no convergence character. However we can prove the following :

THEOREM 1. *In a bicompact space at least one point has convergence character.*

If possible let R be a bicompact space in which no point has convergence character. Then no subset of R flows towards any point. Consider any point p_0 of R. Since R does not flow towards p_0, p_0 has an open neighbourhood U_0 such that $R - U_0$ has the same power as R, and is therefore infinite. Let p_1 be any point of $R - U_0$. Since $R - U_0$ does not flow towards p_1, p_1 has an open neighbourhood U_1 such that $(R - U_0) - U_1$ has the same power as $R - U_0$ and hence is infinite. If for all $\beta < \gamma$ the point p_β and the open neighbourhood U_β of p_β have been defined so that $R - \sum_{\beta<\gamma} U_\beta$ is infinite, let p_γ be any point of $R - \sum_{\beta<\gamma} U_\beta$. Since $R - \sum_{\beta<\gamma} U_\beta$ does not flow towards p_γ, p_γ has an open neighbourhood U_γ such that

$(R - \sum_{\beta<\gamma} U_\beta) - U_\gamma$ has the same power as $R - \sum_{\beta<\gamma} U_\beta$ and hence is infinite. Let $p_{\gamma+1}$ be a point of $R - \sum_{\beta<\gamma} U_\beta$. This process of defining p_β and U_β can be carried out till a stage α where $R - \sum_{\beta<\alpha} U_\beta = P$ is finite. α is clearly a limiting ordinal since from the definition it is clear that if p_γ and U_γ have been defined, $p_{\gamma+1}$ and $U_{\gamma+1}$ can be defined $[R - \sum_{\beta\leqslant\gamma} U_\beta$ being infinite$]$. Let ω_k be the regular initial ordinal cofinal with α, and $[\beta_\gamma]_{\gamma<\omega_k}$ an ω_k series cofinal with α. Consider the set Q of the corresponding points $[p_{\beta_\gamma}]_{\gamma<\omega_k}$. Q has no complete limit points in the set $\sum_{\beta<\alpha} U_\beta$, for let t be any point of $U_\delta\,(\delta < \alpha)$. Then since ω_k is a regular initial ordinal and $[\beta_\gamma]_{\gamma<\omega_k}$ is cofinal with α the subset $(\sum_{\beta_\gamma<\delta} p_{\beta_\gamma})$ of Q has power $< \aleph_k$, the power of Q. But $\sum_{\beta_\gamma\leqslant\delta} p_{\beta_\gamma}$ by definition $= Q \cap (\sum_{\beta\leqslant\delta} U_\beta)$ since every p_β, $\beta > \delta$ belongs to $R - \sum_{\beta\leqslant\delta} U_\beta$. Hence $U_\delta \cap Q$ has power $< \aleph_k$, the power of Q. Therefore since U_δ is an open set containing t, t any point of U_δ $(\delta < \alpha)$ is not a complete limit point of Q, i.e. the complete limit points of Q are contained in P. Let $p^1, p^2, \dots, p^{n+1}$ be the points of P. Then since Q does not flow towards p^1, p^1 has an open neighbourhood V_1 such that $Q - V_1$ has the same power $\aleph_k$, as Q. Since $Q - V_1$ does not flow towards p^2 an open set V_2 containing p^2 can be found such that $(Q - V_1) - V_2$ has power $\aleph_k$. At the nth stage $Q - \sum_{r=1}^{n-1} V_r$ will have power $\aleph_k$ and since this set does not flow towards p^n, an open neighbourhood V_n of p^n can be found such that $(Q - \sum_{r=1}^{n-1} V_r) - V_n$ also has power $\aleph_k$. Then p^{n+1} is the sole complete limit point of $Q - \sum_{r=1}^{n} V_r$. For, since each V_r is open and contains p^r, no point of P other than p^{n+1} is a limit point of this set. Also since $Q - \sum_{r=1}^{n} V_r$ is a subset of Q having the same power as Q, any complete limit point of the former set is a complete limit point of Q also, and hence does not belong to $\sum_{\beta<\alpha} U_\beta$ as shown

above. But since R is bicompact every set has a complete limit point. Hence p^{n+1} is the sole complete limit point of $Q - \sum_{r=1}^{n} V_r$ $(= X)$. Hence X flows towards p^{n+1}, for consider any open neighbourhood U of p^{n+1}. $R - U$ is closed and therefore is bicompact Hence if $X \cap (R - U)$ has power equal to that of X, this set will have a complete limit point in $R - U$ which will be a complete limit point of Q also. This is clearly impossible. Hence $(R - U) \cap X$ has power less than $\aleph_k$ for all neighbourhoods U of p^{n+1}, i.e. X flows towards p^{n+1}. This contradicts the hypothesis. Hence the assumption that no point of R has convergence character is wrong This also leads to the following result.

THEOREM 2 *Every bicompact space R has a weakening under which R is bicompact and every point has a covergence character.*

Let R be any bicompact space. Consider the topology of R in which the open sets are all sets of the form $V \cup F$ where V is any set which is open under the given topology of R and F any set no point of which has convergence character under the given topology of R. The postulates for a basis of open sets is clearly satisfied Also every point of R which had no convergence character under the given topology is isolated under the new topoglogy, while the neighbourhood system of any point which had convergence character under the given topology is unaltered under the new topology. Hence every point has convergence character under the new topology which is clearly weaker than the given topology.

To prove that R is bicompact under the new topology it is clearly sufficient to show that every set has a complete limit point under the new topology, and this is equivalent to : every infinite set has a complete limit point under the given topology of R, which has convergence character. Let L denote the set of points of the given space R which have convergence character. If possible let X be an infinite set no complete limit point of which belongs to L. Let $\aleph_k$ denote the power of X. Then every point p of L has

an (open) neighbourhood U_p such that $U_p \cap X$ has power $< \aleph_k$. The set $(R - \sum_{p \in L} U_p) = M$ is closed and does not contain any point of L. Hence this set is finite since otherwise it would be a bicompact space no point of which has convergence character, which has been shown above to be impossible. Let $m_1, m_2, \dots m_n$ be the points of M. Then since X does not flow towards m_1, m_1 has an open neighbourhood U_1 such that $X - U_1$ has power $\aleph_k$; since $X - U_1$ does not flow towards m_2, m_2 has an open neighbourhood U_2 such that $(X - U_1) - U_2$ has power $\aleph_k$....At the nth stage m_n has an open neighbourhood U_n such that $X - \sum_{r=1}^{n} U_r$ has power $\aleph_k$. This set $X - \sum_{r=1}^{n} U_r$ therefore has a complete limit in $R - \sum_{r=1}^{n} U_r$ since the latter set is bicompact. Also the open sets $\{ U_p \}_{p \in L}$ form an open covering of $R - \sum_{r=1}^{n} U_r$ from which a finite subcovering can be selected Let $U_{p_1}, U_{p_2} \dots U_{p_n}$ be this finite subcovering. Then by definition of the sets $\{ U_p \}$ each set $U_{p_r} \cap [X - (\sum_{r=1}^{n} U_r)]$, being a subset of $U_{p_r} \cap X$, has power $< \aleph_k$. Hence their union, i.e. $X - \sum_{r=1}^{n} U_r$, has power $< \aleph_k$. This contradicts the definition of U_n. Hence the assumption that X has no complete limit point in L, is wrong.

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Ramanujan Institute of Mathematics (Karaikudi)
Madras

ON THE CESÁRO SUMMABILITY OF A CLASS OF FUNCTIONS

By K. PADMAVALLY

[Received October 16, 1952]

1. Introduction. C. T. Rajagopal [4] has proved the following two theorems deducing the second theorem from the first and presenting it as an integral analogue of a theorem of M. S. Macphail [3, Theorem I].

THEOREM A. *Let* $f(x)$ *be integrable over the interval* $(0, w)$, $w > 0$, *and let*

$$f(x+w) = f(x), \quad \frac{1}{w}\int_0^w f(x)\,dx = 0. \tag{1}$$

Then $f(x).\ a(x)$, *where* $a(x)$ *is a polynomial of degree* $k-1$, *is summable* (C, k) *to* 0.

THEOREM B. *Let* $f(x)$ *and* $a(x)$ *be as in Theorem* A. *Let functions* $F_i(x)$ *and constants* c_i , $i = 1,2,3,...$ *be defined as follows* :

$$f_1(x) = \int_0^x f(t)\,dt,\ c_1 = \frac{1}{w}\int_0^w f_1(t)\,dt,\ F_1(x) = f_1(x) - c_1,$$

$$f_2(x) = \int_0^x F_1(t)\,dt,\ c_2 = \frac{1}{w}\int_0^w f_2(t)\,dt,\ F_2(x) = f_2(x) - c_2, \text{ etc.}$$

so that all the functions $F_i(x)$ *satisfy conditions exactly similar to* (1). *Then*

$$\int_0^\infty f(t)\,a(t)\,dt = \sum_{i=0}^{k-1} (-1)^i c_{i+1}\,a^{(i)}(0), \quad (C, k).$$

Rajagopal has also remarked (without proof) that *if* $a(x)$ *in Theorem* B *is not a polynomial but satisfies the condition*

$$\int_0^\infty F_k(x)\,a^{(k)}(x)\,dx \text{ is convergent,} \tag{3}$$

then

$$\int_0^c f(t)\, a(t)dt = \sum_{i=0}^{k-1} (-1)^i c_{i+1} a^{(i)}(0) + (-1)^k \int_0^\infty F_k(x) a^{(k)}(x)dx, \ (C,k).$$

L. S. Bosanquet [1] has pointed out that *for the truth of the above remark,* (3) *alone is insufficient.* I prove here that (3) *is, however, sufficient when we supply a plausible omission by inserting the supplementary condition* :

$$a^{(k)}(x) \to 0, \quad as \quad x \to \infty.$$

My proof leads to generalizations of the remark as well as of Theorems A, B by showing that the conditions (1) which these theorems assume can be suitably relaxed.

2. Lemmas. The following lemmas are required in the proofs of my theorems given in § 3. In these lemmas $I^n[\phi(x)]$ *denotes the* n^{th} *integral of* $\phi(x)$ *with base* $x = 0$ *and* $I_0 [\phi(x)] = \phi(x)$.

LEMMA 1. *If the* k^{th} *derivatives of* $a(x)$, $g(x)$, *exist and are integrable over any finite interval of* $x \geqslant 0$, *then*

$$I_k[g^{(k)}(x) . a(x)] = a \text{ polynomial in } x \text{ of degree } k-1 +$$

$$+ \sum_{i=0}^{k} (-1)^i \binom{k}{i} I_i [g(x) a^{(i)}(x)]. \quad (4)$$

The result clearly holds for $k = 1$. Assuming that the result holds when k is replaced by any positive integer $r \leqslant k - 1$, we shall prove that the result holds also when k is replaced by $r + 1$. In our proof we shall denote by $P_n(x)$ any polynomial of degree n, regardless of the actual values of its coefficients. By definition,

$$I_{r+1}[g^{r+1}(x) a(x)] = I_1 [I_r \{g^{r+1}(x) a(x)\}]$$

$$= I_1 \left[P_{r-1}(x) + \sum_{i=0}^{r} (-1)^i \binom{r}{i} I_i \{ g^{(1)}(x) a^{(r)}(x) \} \right],$$

if we assume (4) for $k = r$ and replace $g(x)$ there by $g^{(1)}(x)$;

$$= P_r(x) + \sum_{i=0}^{r} (-1)^i \binom{r}{i} I_{i+1} [g^{(1)}(x) a^{(i)}(x)]$$

$$=P_r(x)+\sum_{i=0}^{r}(-1)^i\binom{r}{i}I_i\{g(x)a^{(i)}(x)-g(0)a^{(i)}(0)-$$

$$-I_1[g(x)a^{(i+1)}(x)]\}$$

$$=P_r(x)+\sum_{i=0}(-1)^i\binom{r}{i}I_i[g(x)a^i(x)]+$$

$$+\sum_{i=0}^{r}(-1)^{i+1}\binom{r}{i}I_{i+1}[g(x)a^{(i+1)}(x)]$$

$$=P_r(x)+\sum_{i=0}^{r}(-1)^i\binom{r}{i}I_i[g(x)a^{(i)}(x)]+$$

$$+\sum_{i=1}^{r+1}(-1)^i\binom{r}{i-1}I_i[g(x)a^{(i)}(x)]$$

$$=P_r(x)+\sum_{=0}^{r+1}(-1)^i\binom{r+1}{i}I_i[g(x)a^{(i)}(x)].$$

We now complete the proof by induction.

Lemma 2. *If $b(x)$ is integrable over any finite interval of $x \geqslant 0$ and if*

$$b(x) = o(x^s),\ s \geqslant 0,\ x \to \infty,$$

then

$$I_n[b(x)] = o(x^{n+s}),\ x \to \infty.$$

This is a particular case of a well-known theorem concerning ' integral resultants ' [2, Theorem 42].

3. Theorems for functions

Theorem I. *Under the conditions of Lemma* 1 *and the additional conditions*

$$g(x) = O(1),\ a^{(k)}(x) = o(1),\ x \to \infty,$$

we have

$$g^{(k)}(x)\,a(x) \to 0. \quad (C, k).$$

Proof. In Lemma 2, take $n = i$, $0 \leqslant i \leqslant k$, $b(x) = g(x)a^{(i)}(x)$. Then by hypothesis,

$$b(x) = O(1).o(x^{k-i}), x \to \infty,$$

and the conclusion of Lemma 2 assumes the form

$$I_i[g(x)a^{(i)}(x)] = o(x^{i+k-i}), x \to \infty, 0 < i < k. \tag{5}$$

The required result is

$$I_k[g^{(k)}(x)a(x)] = o(x^k), x \to \infty,$$

and this follows at once from (4) when we use (5) in the right hand member of (4).

THEOREM II. *If the kth derivative of $G(t)$ as well as $a(t)$ exist and are integrable in $x > 0$, and if*

$$\int_0^x G(t)\,dt = O(1), a^{(k)}(x) = o(1), x \to \infty, \tag{6}$$

then

$$\int_0^x G^{(k)}(t)\,a(t)\,dt - (-1)^k \int_0^x G(t)\,a^{(k)}(t)\,dt \to \sum_{i=0}^{k-1} (-1)^{i+1} G^{(k-1-i)}(0)\,a^{(i)}(0),\ (C,k).$$

PROOF. By repeated integrations by parts,

$$\int_0^x G^{(k)}(t)\,a(t)\,dt = [G^{(k-1)}(t)\,a(t)]_0^x - [G^{(k-2)}(t)\,a^{(i)}(t)]_0^x + \\ + \dots + (-1)^{k-1}[G(t)\,a^{(k-1)}(t)]_0^x + \\ + (-1)^k \int_0^x G(t)\,a^{(k)}(t)\,dt,$$

or

$$\int_0^x G^{(k)}(t)\,a(t)\,dt - (-1)^k \int_0^x G(t)\,a^{(k)}(t)\,dt$$

$$= \sum_{i=0}^{k-1} (-1)^i\,[a^{(i)}(t)\,G^{(k-1-i)}(t)]_0^x$$

$$= -\left[\sum_{i=0}^{k-1} (-1)^i a^{(i)}(0)\,G^{(k-1-i)}(0)\right] + \\ + \left[\sum_{i=0}^{k-1} (-1)^i a^{(i)}(x)\,G^{(k-1-i)}(x)\right]. \tag{7}$$

The first term in (7) is a constant summable (C, k) to its own value. To complete the proof we have to show that the second term in (7) is summable (C, k) to 0. This follows from the fact that

$$a^{(i)}(x)\, G^{(k-1-i)}(x) \to 0, \quad (C, k-i),\; 0 \leqslant i \leqslant k-1,$$

which is established by writing the hypothesis (6) in the form

$$g(x) = \int_0^x G(t)\, dt = O(1),\; a^{(k-i)}(x) = o(1), \text{ where } a(x) = a^{(0)}(x),$$

and then appealing to Theorem I with $a(x)$ in place of $a(x)$ and $k-i$ in place of k.

Remarks on Theorems I, II. (i) Theorem A of §1 follows from Theorem I when in the latter theorem, we take

$$g(x) = F_k(x) = I_k[f(x)] - c_1 \frac{x^{k-1}}{(k-1)!} - c_2 \frac{x^{k-2}}{(k-2)!} - \dots - c_k, \quad (8)$$

where $f(x)$ is as in Theorem A and $F_k(x)$, $c_1, c_2, \dots c_k$ are defined by (2).

(ii) The generalization of Theorem B in Rajagopal's remark, as amended by me in §1, may be obtained from Theorem II by taking

$$G(x) = F_k(x) = I_k[f(x)] - c_1 \frac{x^{k-1}}{(k-1)!} - c_2 \frac{x^{k-2}}{(k-2)!} - \dots - c_k, \quad (9)$$

where $f(x)$, $F_k(x)$, $c_1, c_2, \dots c_k$ are as in (1) and (2), with the result that, for $0 \leqslant i \leqslant k-1$,

$$G^{(k-1-i)}(x) = I_{i+1}[f(x)] - c_1 \frac{x^i}{i!} - c_2 \frac{x^{i-1}}{(i-1)!} - \dots - c_{i+1},$$

$$G^{(k-1-i)}(0) = -c_{i+1}.$$

(iii) A consideration of the particular cases of Theorems I and II represented by Theorems A and B shows that in the conclusions of Theorems I and II, 'summability (C, k)' cannot in general be replaced by 'summability $(C, k-1)$'.

There is a formal generalization of Theorem II which runs as follows : *If r is a non-negative integer such that $G^{(k+r)}(x)$, $a^{(k+r)}(x)$ exist and are integrable over any finite interval of $x > 0$, and if*

$$\int_0^x G^{(r)}(t)\,dt = O(1),\ a^{(k)}(x) = o(1),\ x \to \infty, \tag{10}$$

$$\int_0^x G(t)\,dt = O(1),\ a^{(r+k)}(x) = o(1),\ x \to \infty, \tag{11}$$

then

$$\int_0^x G^{(r+k)}(t)\,a(t)\,dt - (-1)^{k+r}\int_0^x G(t)\,a^{(k+r)}(t)\,dt \to \sum_{i=0}^{k+r-1} (-1)^{i+1} a^{(i)}(0)\,G^{(k+r-1-i)}(0),\ [C, \max(k, r)].$$

It is clear that, when $r = 0$, (10) and (11) reduce to the single condition (6) and the above result reduces to Theorem II. To prove the above result we start with the following formula obtained by successive integrations by parts

$$\int_0^x G^{(r+k)}(t)\,a(t)\,dt = \sum_{i=0}^{k+r-1} (-1)^{i+1} G^{(k+r-1-i)}(0)\,a^{(i)}(0) + \\ + \sum_{i=0}^{k+r-1} (-1)^{i} a^{(i)}(x)\,G^{(k+r-1-i)}(x) + (-1)^{r+k}\int_0^x G(t)a^{(k+r)}(t)\,dt.$$

We then argue exactly as in the proof of Theorem II, using (10) and Theorem I to prove that

$$a^{(i)}(x)\,G^{(r+k-1-i)}(x) \to 0,\ (C, k-i),\ 0 \leqslant i < k, \tag{12}$$

and using (11) and Theorem I to prove that

$$a^{(i)}(x)\,G^{(r+k-1-i)}(x) \to 0, (C, r+k-i), k \leqslant i \leqslant r+k-1. \tag{13}$$

Combining (12) and (13) we get

$$a^{(i)}(x)\,G^{(k+r-1-i)}(x) \to 0,\ [C, \max(k, r)],\ 0 \leqslant i \leqslant r+k-1,$$

which leads at once to the required result.

It may be pointed out that *in Theorem* II, *any supplementary conditions which ensure*

$$\int_0^k G(t)\, a^{(k)}(t)\, dt \to l(C, r),\ 0 < r < k,$$

modifies the conclusion already obtained to

$$\int_0^x G^{(k)}(t)\, a(t)\, dt \to \sum_{i=0}^{k-1} (-1)^{i+1} a^{(i)}(0)\, G^{(k-1-i)}(0) + (-1)^k . l,\ (C, k).$$

In particular, the supplementary condition, either $a^{(k)}(t)$ *is monotonic, or* $G(x) = O(1)$ *as* $x \to \infty$ *and*

$$\int_0^x | a^{(k)}(t) |\, dt < \infty\ ,$$

suffices to establish that l *exists as the convergent integral*

$$\int_0^\infty G(t)\, a^{(k)}(t)\, dt.$$

The particular case of the above result with the choice of $G(x)$ in (9) which automatically secures the conditions

$$\int_0^x G(t)\, dt = O(1),\ \ G(x) = O(1),\ x \to \infty,$$

is mentioned by Rajagopal [4, p. 25, footnote].

4. Theorems for sequences. For the sake of completeness I shall state without proof the analogues of the main results proved above for functions of a positive integral variable n.

Notation. For any function $\phi(n)$ of the variable n, Σ denotes summation for values 0 to n of the variable, i.e.

$$\Sigma[\phi(n)] = \phi(0) + \phi(1) + \ldots + \phi(n),$$
$$\Sigma_k[\phi(n)] = \Sigma\, \Sigma_{k-1}[\phi(n)],\ k = 1, 2, \ldots, \Sigma_1 = \Sigma\, .$$

The repeated sum $\Sigma_k[\phi(n)]$ is to be distinguished from any repeated sum $S_k[\phi(n)] = \Sigma_k[\phi(n)] +$ a polynomial of degree $k - 1$. Further

$$\Delta\phi(n) = \phi(n+1) - \phi(n),\ \Delta^k \phi(n) = \Delta\, \Delta^{k-1} \phi(n),\ \Delta^1 = \Delta.$$

Lemma I′. $\Sigma_k[\,\Delta^k g(n)\, a(n)] =$ *a polynomial of degree* $k - 1 +$

$$+ \sum_{i=0}^{k} (-1)^i \Sigma_i[\, g(n+k) . \Delta^i a(n+k-i)].$$

There is an obvious analogue of Lemma 2 following from a known theorem on the resultant of two functions of n [2, Theorem 41].

THEOREM I′. *If*

$$g(n) = O(1), \ \Delta^k a(n) = o(1), \ n \to \infty,$$

then

$$\Delta^k g(n).a(n) \to 0, \ (C, k).$$

THEOREM II′. *If*

$$S_{k+1}[f(n)] = O(1), \Delta^k a(n) = o(1), \ n \to \infty,$$

then

$$\sum_1^N f(n)\, a(n) - (-1)^k \sum_1^N S_k[f(n)].\, \Delta^k a(n) \to$$

$$\sum_{i=0}^{k-1} (-1)^i \Delta^i a(0) \{ S_i[f(0)] - S_{i+1}[f(0)] \} \ (C, k), \ N \to \infty.$$

Macphail's theorem is a special case of Theorem II′ where

$$S_k[f(n)] = F_k(n).$$

I wish to thank Prof. C. T. Rajagopal for his help in preparing this note.

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Ramanujan Institute of Mathematics (Karaikudi)
Madras

EIGENFUNCTIONS ON RIEMANNIAN MANIFOLDS

By S. MINAKSHISUNDARAM

[Received October 28, 1953]

1. In this note I give an alternative proof of some of the properties of the eigenfunctions of the Laplace operator on Riemannian manifolds, which Pleijel and I proved some time back [2]. Instead of using Carleman's method, I use the heat equation as I did in the case of the Euclidean space [1].

A parametrix for the heat equation was defined by us [2], using which the fundamental solution is constructed, following Yosida [3]. For any two points P, Q in the manifold, if $U(P, Q\,;\,t)$ stands for the fundamental solution of the heat equation, we note the following properties of the solution :

$$\text{(I)}\quad U(P, Q\,;\,t) = \Sigma\omega_n(P)\,\omega_n(Q)\,e^{-\lambda_n t},\quad t > 0, \tag{1}$$

where λ_n and ω_n are the eigenvalues and the corresponding eigenfunctions of the Laplace operator, so that $U(P, Q\,;\,t) - \frac{1}{V}$ tends to zero exponentially as $t \to \infty$, V being the volume of the manifold.

(II) If P does not coincide with Q, then however large $G > 0$ may be,

$$U(P, Q\,;\,t) = O(t^G),\quad \text{as}\quad t \to 0. \tag{2}$$

(III) If P coincides with Q, an asymptotic development of $U(P, Q\,;\,t)$ in the neighbourhood of $t=0$ is possible in the form,

$$U(P, P\,;\,t) \sim \frac{1}{(2\sqrt{\pi})^k}\,t^{-k/2}(1 + u_1 t + u_2 t^2 + \ldots), \tag{3}$$

where k is the dimension of the manifold and u_ν depend on P only.

From (1) we have

$$\sum_{\lambda_n>0} \omega_n(P)\,\omega_n(Q)\,e^{-\lambda_n t} = U(P, Q; t) - \frac{1}{V}, \tag{4}$$

$$\sum_{\lambda_n>0} \omega_n^2\,e^{-\lambda_n t} = U(P, P; t) - \frac{1}{V}, \tag{5}$$

$$\sum_{\lambda_n>0} e^{-\lambda_n t} = \int_R U(P, P; t)\,dP - 1, \tag{6}$$

where dP stands for the element of volume in the manifold R. From (4), (5) and (6) we deduce that if $R(s) > 0$, and sufficiently large,

$$\begin{aligned}\Gamma(s)\sum \omega_n(P)\,\omega_n(Q)\,\lambda_n^{-s} &= \int_0^\infty \sum \omega_n(P)\,\omega_n(Q)\,e^{-\lambda_n t}\,t^{s-1}\,dt\\ &= \int_0^1 + \int_1^\infty\\ &= \int_0^1 U(P, Q; t)\,t^{s-1}\,dt - \frac{1}{Vs} +\\ &\quad + \int_1^\infty \sum \omega_n(P)\,\omega_n(Q)\,e^{-\lambda_n t}\,t^{s-1}\,dt. \qquad (4')\end{aligned}$$

$$\begin{aligned}\Gamma(s)\sum \omega_n^2\,\lambda_n^{-s} &= \int_0^1 U(P, P; t)\,t^{s-1}\,dt - \frac{1}{Vs} +\\ &\quad + \int_1^\infty \sum \omega_n^2\,e^{-\lambda_n t}\,t^{s-1}\,dt. \qquad (5')\end{aligned}$$

$$\begin{aligned}\Gamma(s)\sum \lambda_n^{-s} &= \int_0^1 \left(\int_R U(P, P; t)\,dP\right) t^{s-1}\,dt - \frac{1}{s} +\\ &\quad + \int_1^\infty \sum e^{-\lambda_n t}\,t^{s-1}\,dt. \qquad (6')\end{aligned}$$

Using the results (2) and (3) in the integrals over (0, 1) on the right side of (4′), (5′) and (6′) we obtain the properties of the eigenfunctions and eigenvalues proved by us. If we use (3) in (5) and (6) and apply Hardy-Littlewood Tauberian theorem we obtain the asymptotic distribution of the eigenvalues and eigenfunctions.

2. Now we have only to construct the fundamental solution and prove its properties. We assume the results of § 1 of our paper [2],

where a parametrix is defined, and briefly sketch some of them here, leading to the parametrix of the heat equation.

Let R be a compact connected analytic manifold with a metric

$$dr^2 = g_{ij}\, dx^i\, dx^j, \quad i, j = 1, 2, \ldots, k.$$

We denote by Δ the Laplace operator

$$\Delta u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, g^{ij} \frac{\partial u}{\partial x^j} \right),$$

and by δ the parabolic operator

$$\delta u = \Delta u - \frac{\partial u}{\partial t}.$$

A parametrix of the parabolic equation

$$\delta u = 0,$$

in the neighbourhood of a point P in R, will be given by

$$H_n(P, Q; t - s), \quad \text{for } 0 \leqslant s < t,$$

$$H_n(P, Q; t - s) = \frac{1}{(2\sqrt{\pi})^k} \frac{1}{(t - s)^{k/2}} e^{-r^2/4(t-s)} \times$$
$$\times \left\{ U_0(P, Q) + (t - s)\, U_1(P, Q) + \ldots + (t - s)^n\, U_n(P, Q) \right\},$$

r being the geodesic distance between P and Q and $u_\nu(P, Q)$ being defined recursively by the differential equations [2, p. 244],

$$r \frac{du_\nu}{dr} + \left(\frac{r}{2} \log \sqrt{g} + \nu \right) u_\nu = u_{\nu-1},$$
$$u_{-1} \equiv 0, \quad u_0(P, P) = 1.$$

If the manifold is analytic we can choose n as large as we please. Analyticity is a regularity condition on the manifold, which is sufficient for our purpose, though the condition can be considerably relaxed. A parametrix in the large will be

$$\tilde{H}_n(P, Q; t - s) = \eta_\epsilon(r_{PQ})\, H_n(P, Q; t - s), \tag{8}$$

where $\eta_\epsilon(x)$ is a continuous function of x differentiable sufficiently often so that

$$\left.\begin{array}{l} \eta_\epsilon(x) = \begin{cases} 1, & 0 \leqslant x \leqslant \epsilon/2, \\ 0, & \epsilon \leqslant x, \end{cases} \\ |\eta_\epsilon(x)| \leqslant 1, \\ \left|\dfrac{d}{dx}\eta_\epsilon(x)\right| = O\left(\dfrac{1}{\epsilon}\right), \quad \left|\dfrac{d^2}{dx^2}\eta_\epsilon\right| = O\left(\dfrac{1}{\epsilon^2}\right). \end{array}\right\} \tag{9}$$

We observe that

$$\delta_{tQ} H_n = \frac{1}{(2\sqrt{\pi})^k} \Delta_Q u_n (t-s)^{n-k/2} e^{-r^2/4(t-s)}, \tag{10}$$

and we set

$$K_n(P, Q; t, s) = \delta_{tQ} \widetilde{H}_n. \tag{11}$$

Then, by virtue of (9) and (10), there exists a positive constant M so that for all P and Q in R,

$$\left.\begin{array}{r} |K_n| \leqslant M(t-s)^{n-k/2}, \\ \left|\int_R K_n\, dQ\right| \leqslant M V (t-s)^{n-k/2}, \end{array}\right\} \tag{12}$$

and if we set for every $m = 2, 3, \ldots$

$$K_n^m(P, Q; t, s) = \int_s^t \int_R K_n^{m-1}(P, U, \tau - s)\; K_n(U, Q, t-\tau)\, dU\, d\tau$$

$$K^1{}_n = K_n,$$

we observe by induction that

$$|K_n^m| \leqslant \frac{M^m V^{m-1}\{\Gamma(n-k/2+1)\}^m}{\Gamma(mn-k/2+1)} (t-s)^{m(n-k/2)+m-1}. \tag{13}$$

A fundamental solution of the heat equation $\delta u = 0$, will be

$$U(P, Q; t, s) = \widetilde{H}_n - \int_s^t \int_R V_n(P, U, \tau - s)\, \widetilde{H}_n(U, Q; t-\tau)\, dU\, d\tau, \tag{14}$$

where

$$V_n(P, Q; t, s) = \sum_{m=1}^{\infty} (-)^{m-1} K_n^m(P, Q; t, s), \tag{15}$$

the series in (15) converging absolutely and uniformly on account of (13).

From (12) and (13) we observe that if $0 \leq s < t \leq 1$,

$$V_n(P, Q; t, s) = O\{(t-s)^{n-k/2}\}, \tag{16}$$

uniformly for P, Q in R. Now we set

$$U(P, Q; t) = U(P, Q; t, 0), \tag{17}$$

and if P and Q are distinct points in R, we choose ϵ so small that the geodesic spheres of radius ϵ round P and Q have no points in common, so that

$$\tilde{H}_n(P, Q; t-s) \equiv 0,$$

and

$$\begin{aligned} U(P, Q; t) &= -\int_0^t \int_R V_n(P, U; \tau, 0)\, \tilde{H}_n(U, Q, t-\tau)\, dU\, d\tau \\ &= O \int_0^t \int_R e^{-r^2/4(t-\tau)} (t-\tau)^{-k/2} \tau^{n-k/2} \eta_\epsilon(r)\, dU\, d\tau \\ &= O \int_0^t \int_0^\infty e^{-r^2/4(t-\tau)} (t-\tau)^{-k/2} \tau^{n-k/2} r^{k-1}\, dr\, d\tau \\ &= O(t^{n-k/2+1}). \end{aligned} \tag{18}$$

In this chain of equalities, normal coordinates are used in the last but one, from which the last is deduced using the transformation $\tau = v$, $r^2/4\,(t-\tau) = u$.

On the other hand if P coincides with Q, $H_n = \tilde{H}_n$, and

$$\begin{aligned} U(P, P; t) &= H_n - \int_0^t \int_R V_n\, \tilde{H}_n\, du\, d\tau \\ &= \frac{1}{(2\sqrt{\pi})^k} t^{-k/2} \left\{ 1 + u_1(P, P)\, t + \ldots + u_n(P, P) t^n \right\} + \\ &\qquad + O(t^{n-k/2+1}), \end{aligned} \tag{19}$$

by (18). As n can be as large as we please (2) and (3) will follow from (18) and (19).

3. To prove (1) we first observe that if $f(P)$ is any bounded continuous function defined in R,

$$f(P) = \lim_{t=0} \int_R U(P, Q; t) f(Q)\, dQ$$
$$= \lim_{t=0} \int_R U(Q, P; t) f(Q)\, dQ, \qquad (20)$$

for if we set $U(P, Q; t) = \tilde{H}_n - G_n$, then

$$\lim_{t=0} \int_R G_n f = 0,$$

by virtue of (18), and as $\tilde{H}_n$ is of the nature of the Gaussian kernel,

$$\lim_{t=0} \int_R \tilde{H}_n f = f(P).$$

Secondly we note that

$$\delta_{tQ} U(P, Q; t) = 0, \qquad (21)$$

for

$$\begin{aligned} \delta_{tQ} U &= \delta_{tQ} \tilde{H}_n - \delta_{tQ} \int_0^t \int_R V_n \tilde{H}_n \\ &= K_n - \int_0^t \int_R V_n \Delta_Q \tilde{H}_n + \frac{\partial}{\partial t} \int_0^t \int_R V_n \tilde{H}_n \\ &= K_n - \int_0^t \int_R V_n \Delta_Q \tilde{H}_n + \int_0^t \int_R V_n \frac{\partial}{\partial t} \tilde{H}_n - V_n \\ &= K_n - V_n - \int_0^t \int_R V_n K_n \\ &= 0, \end{aligned}$$

by (15). Thirdly if we set

$$f_m(P, t) = \int_R U(P, Q; t)\, \omega_m(Q)\, dQ,$$

then we observe that $\lim\limits_{t=0} f_m(P, t) = \omega_m(P)$, by (20), and

$$\frac{\partial}{\partial t} f_m(P, t) = \int_R \frac{\partial}{\partial t} U(P, Q; t)\, \omega_m(Q)\, dQ = \int_R \Delta_Q U.\ \omega_m\, dQ$$

$$= \int_R U.\ \Delta_Q\, \omega_m\, dQ = -\lambda_m f_m(P, t),$$

so that

$$f_m(P; t) = \omega_m(P)\, e^{-\lambda_m t}.$$

Hence

$$U(P, Q; t) \sim \Sigma\, \omega_m(P)\, \omega_m(Q)\, e^{-\lambda_m t}.$$

Since the right side is absolutely convergent, (1) will follow from the completeness property of the eigenfunctions.

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Andhra University
Waltair

SOME INFINITE INTEGRALS INVOLVING E-FUNCTIONS

By C. B. RATHIE

[Received July 28, 1952]

1. The object of this paper is to evaluate a few infinite integrals on MacRobert's E-function by means of the following theorems in operational calculus

If $f(p) \doteqdot h(x)$ *and* $p^{-2\nu} h\left(\frac{1}{p}\right) \doteqdot g(x)$, *then* [10]

$$f(p) = 2\, p^{\nu+1} \int_0^\infty t^{-\nu} K_{2\nu}(2\sqrt{pt})\, g(t)\, dt, \tag{1}$$

and if, in addition, $F(p) \doteqdot x^{\frac{1}{2}-\nu-l}\, g(x)$, *then* [3, p. 453]

$$f(p) = p^{\nu-l+\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2}u}\, u^{l-1}\, W_{l,\nu}(u)\, F\left(\frac{p}{u}\right) du, \tag{2}$$

when the integrals converge.

We shall also use the Parseval-Goldstein theorem [2], viz.

If $f(p) \doteqdot h(x)$ *and* $\phi(p) \doteqdot g(x)$, *then*

$$\int_0^\infty x^{-1} f(x)\, g(x)\, dx = \int_0^\infty x^{-1} h(x)\, \phi(x)\, dx. \tag{3}$$

The notation $f(p) \doteqdot h(x)$ means that

$$f(p) = p \int_0^\infty e^{-px}\, h(x)\, dx.$$

The integrals investigated in this paper are quite general and include as particular cases some well-known results.

2. (i) Starting with [5, p. 10]

$$h(x) = x^{\gamma-1}\, {}_2F_1(\alpha, \beta; \delta; -x)$$

$$\doteqdot \frac{\Gamma(\delta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, p^{1-\gamma}\, E(\alpha, \beta, \gamma : \delta : p)$$

$$= f(p),\ R(\gamma) > 0,\ \ R(p) > 0,$$

we have, on term by term interpretation,

$$p^{-2\nu} h\left(\frac{1}{p}\right) \doteqdot \frac{x^{2\nu+\gamma-1}}{\Gamma(2\nu+\gamma)}\ {}_2F_2(\alpha,\ \beta;\ \delta,\ 2\nu+\gamma;\ -x)$$

$$= g(x),\ \ R(2\nu+\gamma) > 0,$$

and

$$x^{\frac{1}{2}-\nu-l}\, g(x) \doteqdot \frac{\Gamma(\nu+\gamma-l+\frac{1}{2})}{\Gamma(2\nu+\gamma)}\, p^{l-\nu-\gamma+\frac{1}{2}} \times$$

$$\times\ {}_3F_2(\alpha, \beta, \nu+\gamma-l+\tfrac{1}{2}; \delta, 2\nu+\gamma; -1/p)$$

$$= F(p),\ R(\nu+\gamma-l+\tfrac{1}{2}) > 0.$$

Hence, from (1) we get

$$E(\alpha, \beta, \gamma: \delta: p) = \frac{2\,\Gamma(\alpha)\Gamma(\beta)\,p^{\nu+\gamma}}{\Gamma(\delta)\Gamma(2\nu+\gamma)}\int_0^\infty t^{\nu+\gamma-1} K_{2\nu}(2\sqrt{pt}) \times$$

$$\times\ {}_2F_2(\alpha, \beta; \delta, 2\nu+\gamma; -t)\,dt, \qquad (4)$$

where $R(\gamma) > 0,\ \ R(2\nu+\gamma) > 0,\ \ R(p) > 0.$

Taking $\nu = \frac{1}{4}$, replacing $2\sqrt{p}$ by p and making a slight change of variable we have

$$E(\alpha, \beta, \gamma: \delta: \tfrac{1}{4}p^2) = \frac{\sqrt{\pi}\,\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\delta)\Gamma(\gamma+\frac{1}{2})}\ (p/2)^{2\gamma-1} p \int_0^\infty e^{-px} x^{2\gamma-1} \times$$

$$\times\ {}_2F_2(\alpha, \beta; \delta, \gamma+\tfrac{1}{2}; -x^2)\,dx,$$

from which it follows that

$$x^{2\gamma-1}\ {}_2F_2(\alpha,\ \beta;\ \delta,\ \gamma+\tfrac{1}{2};\ -x^2)$$

$$\doteqdot \frac{\Gamma(\delta)\Gamma(\gamma+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\alpha)\Gamma(\beta)}\,(2/p)^{2\gamma-1} E(\alpha, \beta, \gamma: \delta: \tfrac{1}{4}p^2),$$

$$R(\gamma) > 0,\ \ R(p) > 0. \qquad (5)$$

If in (4) we take $\gamma = \delta$, $\alpha = \frac{1}{2} - k - m$, $\beta = \frac{1}{2} - k + m$, and use the relation

$$E(\tfrac{1}{2}-k-m,\ \tfrac{1}{2}-k+m\ \cdot\ : p) = \Gamma(\tfrac{1}{2}-k-m)\,\Gamma(\tfrac{1}{2}-k+m) \times$$

$$\times\ p^{-k} e^{\frac{1}{2}p}\, W_{k,m}(p), \qquad (6)$$

we get a result due to Hari Shanker [3, p. 453].

Next applying (2) we get

$$E(\alpha,\beta,\gamma:\delta:p)=\frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\nu+\gamma-l+\tfrac{1}{2})}{\Gamma(\delta)\,\Gamma(2\nu+\gamma)}\times$$

$$\times\int_0^\infty e^{-\frac{1}{2}u}\,u^{\nu+\gamma-3/2}\,W_{l,\nu}(u)\times$$

$$\times\,{}_3F_2(\alpha,\beta,\nu+\gamma-l+\tfrac{1}{2};\delta,2\nu+\gamma;-u/p)\,du, \quad (7)$$

valid for $R(\gamma)>0$, $R(2\nu+\gamma)>0$, $R(p)>0$.

When $l=\nu+\frac{1}{2}$, this gives

$$E(\alpha,\beta,\gamma:\delta:p)=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\delta)\Gamma(2\nu+\gamma)}\int_0^\infty e^{-u}u^{2\nu+\gamma-1}\times$$

$$\times\,{}_3F_2(\alpha,\beta,\gamma;\delta,2\nu+\gamma;-u/p)\,du,\quad R(2\nu+\gamma)>0,\ R(p)>0,$$

from which we derive that

$$x^{\lambda-1}\,{}_3F_2(\alpha,\beta,\gamma;\delta,\lambda;-x)$$

$$\doteqdot\frac{\Gamma(\delta)\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}p^{1-\lambda}E(\alpha,\beta,\gamma:\delta:p),\quad R(\lambda)>0,\ R(p)>0.\quad (8)$$

When $\gamma=\delta$, (7) yields by virtue of (6) an integral due to Erdélyi [1].

(ii) Now taking the relation (8) and proceeding in a similar manner the following results are easily obtained.

$$E(\alpha,\beta,\gamma:\delta:p)=\frac{2\,\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\delta)\Gamma(\lambda)\Gamma(2\nu+\lambda)}p^{\lambda+\nu}\times$$

$$\times\int_0^\infty t^{\lambda+\nu-1}K_{2\nu}(2\sqrt{pt})\,{}_3F_3(\alpha,\beta,\gamma;\delta,\lambda,2\nu+\lambda;-t)\,dt,$$

$$R(\lambda)>0,\ R(2\nu+\lambda)>0,\ R(p)>0.\quad (9)$$

$$x^{2\lambda-1}\,{}_3F_3(\alpha,\beta,\gamma;\delta,\lambda,\lambda+\tfrac{1}{2};-x^2)$$

$$\doteqdot\frac{\Gamma(\delta)\Gamma(2\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}p^{1-2\lambda}E(\alpha,\beta,\gamma:\delta:\tfrac{1}{4}p^2),\ R(\lambda)>0,\ R(p)>0.\quad (10)$$

$$E(\alpha,\beta,\gamma:\delta:p)=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\nu+\lambda-l+\tfrac{1}{2})}{\Gamma(\delta)\Gamma(\lambda)\Gamma(2\nu+\lambda)}\times$$

$$\times\int_0^\infty e^{-\frac{1}{2}u}\,u^{\nu+\lambda-3/2}\,W_{l,\nu}(u)\times$$

$$\times\ {}_4F_3(\alpha, \beta, \gamma, \nu+\lambda-l+\tfrac{1}{2}; \delta, \lambda, 2\nu+\lambda; -u/p)\,du,$$

$$R(\lambda) > 0, R(2\nu+\lambda) > 0, R(p) > 0. \qquad (11)$$

(iii) Now starting with (10)

$$h(x) = x^{2\lambda-1}\,{}_3F_3(\alpha, \beta, \gamma; \delta, \lambda, \lambda+\tfrac{1}{2}; -x^2)$$

$$\doteqdot \frac{\Gamma(\delta)\Gamma(2\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} p^{1-2\lambda} E(\alpha, \beta, \gamma : \delta : \tfrac{1}{4}p^2)$$

$$= f(p), \quad R(\lambda) > 0, R(p) > 0,$$

we find that

$$p^{-2\nu} h(1/p) \doteqdot \frac{x^{2\nu+2\lambda-1}}{\Gamma(2\nu+2\lambda)} \times$$

$$\times\ {}_3F_5(\alpha, \beta, \gamma; \delta, \nu+\lambda, \nu+\lambda+\tfrac{1}{2}, \lambda, \lambda+\tfrac{1}{2}; -\tfrac{1}{4}x^2)$$

$$= g(x)\ ,\ R(\nu+\lambda) > 0,$$

and that

$$x^{\frac{1}{2}-\nu-l}\, g(x) \doteqdot \frac{\Gamma(\nu+2\lambda-l+\frac{1}{2})}{\Gamma(2\nu+2\lambda)} p^{\frac{1}{2}+l-\nu-2\lambda} \times$$

$$\times\ {}_5F_5\left\{\begin{matrix} \alpha, \beta, \gamma, \frac{1}{4}+\lambda+\frac{1}{2}(\nu-l), \frac{3}{4}+\lambda+\frac{1}{2}(\nu-l); \\ \delta, \nu+\lambda, \nu+\lambda+\frac{1}{2}, \lambda, \lambda+\frac{1}{2}; \end{matrix}\ -1/p^2\right\}$$

$$= F(p), R(\nu+2\lambda-l+\tfrac{1}{2}) > 0.$$

Hence, from (2) we get

$$E(\alpha, \beta, \gamma : \delta : \tfrac{1}{4}p^2) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\nu+2\lambda-l+\frac{1}{2})}{\Gamma(\delta)\Gamma(2\lambda)\Gamma(2\nu+2\lambda)} \times$$

$$\times \int_0^\infty e^{-\frac{1}{2}u} u^{2\lambda+\nu-3/2} W_{l,\nu}(u) \times$$

$$\times\ {}_5F_5\left\{\begin{matrix} \alpha, \beta, \gamma, \frac{1}{4}+\lambda+\frac{1}{2}(\nu-l), \frac{3}{4}+\lambda+\frac{1}{2}(\nu-l); \\ \delta, \nu+\lambda, \nu+\lambda+\frac{1}{2}, \lambda, \lambda+\frac{1}{2}; \end{matrix}\ -u^2/p^2\right\} du, \qquad (12)$$

valid for $R(\lambda) > 0$, $R(\nu+\lambda) > 0$, $R(p) > 0$.

When $\gamma = \delta$, this yields by virtue of (6) a known result [3, p. 454].

(iv) Taking [7, p. 255]

$$h(x) = x^{\gamma-1}\, E(a_1, \ldots, a_r, 2\nu+\gamma : \beta_1, \ldots \beta_s, \nu+\gamma-l+\tfrac{1}{2} : 1/x)$$
$$\doteqdot p^{1-\gamma}\, E(a_1, \ldots, a_r, \gamma, 2\nu+\gamma : \beta_1, \ldots, \beta_s, \nu+\gamma-l+\tfrac{1}{2} : p)$$
$$= f(p),\ R(\gamma) > 0,\ R(p) > 0,$$

we find that

$$p^{-2\nu} h(1/p) \doteqdot x^{2\nu+\gamma-1}\, E(a_1, \ldots, a_r : \beta_1, \ldots, \beta_s, \nu+\gamma-l+\tfrac{1}{2} : 1/x)$$
$$= g(x),\ R(2\nu+\gamma) > 0,$$

and

$$x^{\frac{1}{2}-\nu-l}\, g(x) \doteqdot p^{\frac{1}{2}+l-\nu-\gamma}\, E(a_1, \ldots, a_r : \beta_1, \ldots, \beta_s : p)$$
$$= F(p),\ R(\nu+\gamma-l+\tfrac{1}{2}) > 0.$$

Thus on applying (1) we get a result due to MacRobert [7, p. 258] while (2) yields

$$E(a_1, \ldots, a_r, \gamma, 2\nu+\gamma : \beta_1, \ldots, \beta_s, \nu+\gamma-l+\tfrac{1}{2} : p)$$
$$= \int_0^\infty e^{-\frac{1}{2}u}\, u^{\nu+\gamma-3/2}\, W_{l,\nu}(u)\, E(a_1, \ldots, a_r : \beta_1, \ldots, \beta_s : p/u)\, du, \quad (13)$$

valid for $R(\gamma) > 0$, $R(2\nu+\gamma) > 0$, $R(p) > 0$.

A few special cases of (13) are worth mentioning.

(a) Since

$$E(\tfrac{1}{2}-2\mu, \tfrac{1}{2}+2\mu, \tfrac{1}{2} : : x) = \tfrac{1}{2}\pi^{5/2}\sec(2\mu\pi)\sqrt{x}\, H^{(1)}_{2\mu}(\sqrt{x})\, H^{(2)}_{2\mu}(\sqrt{x})$$

we have, on taking $a_1 = \tfrac{1}{2}-2\mu$, $a_2 = \tfrac{1}{2}$, $a_3 = 2\mu+\nu-l+1$, $\beta_1 = 2\mu+2\nu+\tfrac{1}{2}$, $a_4 = \beta_2, \ldots, a_r = \beta_s$, and $\gamma = 2\mu+\tfrac{1}{2}$,

$$H^{(1)}_{2\mu}(\sqrt{p})\, H^{(2)}_{2\mu}(\sqrt{p}) = \frac{2\cos(2\mu\pi)}{\pi^2\sqrt{\pi p}} \int_0^\infty e^{-\frac{1}{2}u}\, u^{2\mu+\nu-1}\, W_{l,\nu}(u) \times$$
$$\times\, E(\tfrac{1}{2}-2\mu, \tfrac{1}{2}, 2\mu+\nu-l+1 : 2\mu+2\nu+\tfrac{1}{2} : p/u)\, du,$$
$$R(\mu) > -\tfrac{1}{4},\ R(2\mu+2\nu+\tfrac{1}{2}) > 0,\ R(p) > 0. \quad (14)$$

Further $\nu = -\mu$ will give us by virtue of (6)

$$H^{(1)}_{2\mu}(\sqrt{p})\, H^{(2)}_{2\mu}(\sqrt{p}) = \frac{2\,\Gamma(\mu-l+1)\, p^{-\frac{1}{2}(\mu+l+\frac{1}{2})}}{\pi\sqrt{\pi}\,\Gamma(2\mu+\tfrac{1}{2})} \int_0^\infty e^{-\frac{1}{2}(u-p/u)} \times$$
$$\times\, u^{\frac{1}{2}(3\mu+l-5/2)}\, W_{l,\mu}(u)\, W_{\frac{1}{2}(\mu+l-\frac{1}{2}),\frac{1}{2}(3\mu-l+\frac{1}{2})}(p/u)\, du,$$
$$R(\mu) > -\tfrac{1}{4},\ R(p) > 0. \quad (15)$$

(b) Again since

$$E(\tfrac{1}{2}-k+m,\ \tfrac{1}{2}-k-m,\ 1::x)$$

$$=\Gamma(\tfrac{1}{2}-k+m)\,\Gamma(\tfrac{1}{2}-k-m)\,(4x)^{\frac{1}{2}-k}\,S_{2k,\,2m}(2\sqrt{x})$$

on taking $\alpha_1=\frac{1}{2}-k+m$, $\alpha_2=\frac{1}{2}-k-m$, $\alpha_3=\nu-l+3/2$, $\beta_1=2\nu+1$, $\alpha_4=\beta_2,\ldots,\ \alpha_r=\beta_s$ and $\gamma=1$, we get

$$S_{2k,\,2m}(2\sqrt{p})=\frac{(4p)^{k-\frac{1}{2}}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}\int_0^\infty e^{-\frac{1}{2}u}\,u^{\nu-\frac{1}{2}}\,W_{l,\nu}(u)\times$$

$$\times E(\tfrac{1}{2}-k+m,\ \tfrac{1}{2}-k-m,\ \nu-l+3/2:2\nu+1\,.\ p/u)\,du,$$

$$R(\nu)>-\tfrac{1}{2},\ R(p)>0. \qquad (16)$$

If, in this we take $\nu=\frac{1}{2}(m-k-\frac{1}{2})$ and use (6) we have

$$S_{2k,\,2m}(2\sqrt{p})=\frac{\Gamma(\frac{1}{2}m-\frac{1}{2}k-l+5/4)}{\Gamma(\frac{1}{2}-k+m)}\,2^{2k-1}\,p^{\frac{1}{4}(2k-2m-4l-1)}\times$$

$$\times\int_0^\infty e^{-\frac{1}{2}(u-p/u)}\,u^{\frac{1}{4}(6m+2k+4l-9)}\,W_{l,\frac{1}{2}(m-k-\frac{1}{2})}(u)\times$$

$$\times W_{\frac{1}{4}(2m+6k+4l-3),\ \frac{1}{4}(2k+6m-4l+3)}(p/u)\,du, \qquad (17)$$

valid for $R(m-k+\frac{1}{2})>0$, $R(p)>0$.

If in (16) we take $l=\frac{1}{2}-\nu$ and in (17) $2l=m-k+\frac{1}{2}$, we get results given by Meijer [9, p. 630].

(c) If we take $\alpha_1=\frac{1}{2}-k+m$, $\alpha_2=\frac{1}{2}-k-m$, $\alpha_3=\beta_1,\ldots,$ $\alpha_r=\beta_s$, and use (6) we have

$$E(\tfrac{1}{2}-k+m,\tfrac{1}{2}-k-m,\gamma,2\nu+\gamma:\nu+\gamma-l+\tfrac{1}{2}:p)$$

$$=\Gamma(\tfrac{1}{2}-k+m)\Gamma(\tfrac{1}{2}-k-m)\,p^{-k}\int_0^\infty e^{-\frac{1}{2}(u-p/u)}\,u^{k+\nu+\gamma-3/2}\times$$

$$\times W_{l,\nu}(u)\,W_{k,m}(p/u)\,du, \qquad (18)$$

valid for $R(\gamma)>0$, $R(2\nu+\gamma)>0$, $R(p)>0$.

A few integrals given by Meijer [8, p. 20] follow as particular cases of this result. For example, if we take $\nu=\frac{1}{4}$, $l=k+\frac{1}{4}$, $\gamma=\frac{1}{2}-k$, replace p by p^2 and use the relations

$$W_{\frac{1}{2}\nu+\frac{1}{4},\frac{1}{4}}\left(\tfrac{1}{2}z^2\right)=2^{-\frac{1}{2}\nu-\frac{1}{4}}z^{\frac{1}{2}}D_\nu(z)$$

and

$$E\left(\tfrac{1}{2}-k+m,\tfrac{1}{2}-k-m,\tfrac{1}{2}-k,1-k:1-2k:\tfrac{1}{4}z^2\right)$$

$$=\Gamma\left(\tfrac{1}{2}-k+m\right)\Gamma\left(\tfrac{1}{2}-k-m\right)\sqrt{\pi}\,2^{2k}z^{-2k}W_{k,m}(iz)W_{k,m}(-iz)$$

we get

$$W_{k,m}(2ip)W_{k,m}(-2ip)=\frac{2^{-k}}{\sqrt{\pi}}\int_0^\infty \exp\left[-\tfrac{1}{2}(u-p^2/u)\right]u^{-\frac{1}{2}}\times$$

$$\times D_{2k}(\sqrt{2u})\,W_{k,m}(p^2/u)\,du. \quad (19)$$

3. We shall now evaluate an integral with the help of Parseval-Goldstein theorem mentioned above.

Take the well-known operational relations

$$h(x)=e^{-zx}x^{\gamma-1}(1+x)^{\delta-\alpha-\beta}$$

$$\doteqdot\frac{p(p+z)^{-\gamma}}{\Gamma(\alpha+\beta-\delta)}E(\alpha+\beta-\delta,\gamma::p+z)$$

$$=f(p),\ R(\gamma)>0,\ R(z)>0,$$

and

$$\phi(p)=\frac{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)}{\Gamma(\delta)}p\,{}_2F_1(\delta-\alpha,\delta-\beta;\delta;-p)$$

$$\doteqdot\frac{e^{-x}x^{\delta-\alpha-\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}E(\alpha,\beta::x)$$

$$=g(x),\ R(\delta-\alpha)>0,\ R(\delta-\beta)>0.$$

Substituting these values in (3) and evaluating the integral on the right with the help of [6, p. 98]

$$\int_0^\infty e^{-zx}x^{\gamma-1}(1+x)^{\alpha+\beta-\delta}\,{}_2F_1(\alpha,\beta;\delta;-x)\,dx$$

$$=\frac{\Gamma(\delta)z^{-\gamma}}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)}E(\delta-\alpha,\delta-\beta,\gamma:\delta:z),\ R(\gamma)>0,\ R(z)>0,$$

we find that

$$E(\alpha, \beta, \gamma : \delta : z) = \frac{z^{\gamma}}{\Gamma(\delta - \alpha)\,\Gamma(\delta - \beta)\,\Gamma(\alpha + \beta - \delta)} \times$$

$$\times \int_0^{\infty} (x + z)^{-\gamma}\, x^{\delta - \alpha - \beta - 1}\, e^{-x}\, E(\alpha + \beta - \delta, \gamma : : x + z) \times$$

$$\times\, E(\alpha, \beta : : x)\, dx, \quad (20)$$

valid for $R(\delta - \alpha) > 0$, $R(\delta - \beta) > 0$, $R(z) > 0$.

When $\gamma = \delta$, we have

$$E(\alpha, \beta : : z) = \frac{z^{\gamma}}{\Gamma(\gamma - \alpha)\,\Gamma(\gamma - \beta)\,\Gamma(\alpha + \beta - \gamma)} \times$$

$$\times \int_0^{\infty} (x + z)^{-\gamma}\, x^{\gamma - \alpha - \beta - 1}\, e^{-x}\, E(\alpha + \beta - \gamma, \gamma : : x + z) \times$$

$$\times\, E(\alpha, \beta : : x)\, dx, \; R(\gamma - \alpha) > 0, \; R(\gamma - \beta) > 0, \; R(z) > 0. \quad (21)$$

This is equivalent to a known result [4, p. 482].

When $\delta \rightarrow \alpha + \beta$, we get from (20), by virtue of the relation

$$\lim_{\alpha \to 0} \frac{E(\alpha, \beta : : x)}{\Gamma(\alpha)} = \Gamma(\beta),$$

$$E(\alpha, \beta, \gamma : \alpha + \beta : z)$$

$$= \frac{z^{\gamma}\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^{\infty} (x + z)^{-\gamma}\, x^{-1}\, e^{-x}\, E(\alpha, \beta : : x)\, dx,$$

$$R(\alpha) > 0, \; R(\beta) > 0, \; R(z) > 0. \quad (22)$$

My best thanks are due to Dr. R. S. Varma for his guidance and interest in the preparation of this paper.

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Maharana Bhupal College
Udaipur

THE EQUIVALENCE OF TWO CONJECTURES IN THE THEORY OF NUMBERS

By P. T. BATEMAN and S. CHOWLA

[Received December 4, 1953]

The following conjecture has been discussed by Turán [6].

CONJECTURE I. $\sum_{\nu=1}^{n} \frac{\lambda(\nu)}{\nu} > 0$,

for any positive integer n. (λ denotes the Liouville function)

The purpose of this note is to point out that the preceding conjecture is equivalent to the following one.

CONJECTURE II. $\sum_{\nu=1}^{n} \frac{\chi(\nu)}{\nu} > 0$,

for any positive integer n and any real residue-character χ.

The equivalence of these two conjectures can even be put in the following sharper form : *If N is any positive integer, Conjecture* I *is true for all* $n \leqslant N$ *if and only if Conjecture* II *is true for all* $n \leqslant N$ *and all real characters* χ. In one direction this is trivial, since $\lambda(\nu) = (\nu \mid N^*)$ for $\nu = 1, 2, \ldots, N$, where $(x \mid y)$ denotes the Legendre-Jacobi symbol and N^* denotes the smallest positive integer such that $(p \mid N^*) = -1$ for all primes p not exceeding N. (The existence of N^* follows from the quadratic reciprocity law and the Chinese remainder theorem. For example, $2^* = 3$, $3^* = 4^* = 5$, $5^* = \ldots = 10^* = 43$, $11^* = \ldots = 16^* = 67$, $17^* = \ldots = 40^* = 163$). The other half of this sharper form of the equivalence is contained in the theorem below, which was communicated to us by A. B. Showalter. To simplify the statement of the theorem let us call an arithmetic function *admissible* if it satisfies the following three conditions : (i) it is completely multiplicative, that is, $\chi(mn) = \chi(m)\chi(n)$, for any positive integers m and n, (ii) it is not

identically zero, that is $\chi(1) = 1$, (iii) if p is a prime number, $\chi(p)$ is 1, 0, or -1. Then Showalter's theorem is as follows:

THEOREM. *If* $\sum_{\nu=1}^{n} \lambda(\nu)/\nu > 0$ *for all* $n < N$ *and if* χ *is an admissible arithmetic function, then*

$$\sum_{\nu=1}^{n} \frac{\chi(\nu)}{\nu} > \sum_{\nu=1}^{n} \frac{\lambda(\nu)}{\nu}, \quad \textit{for all } n < N,$$

with equality throughout only if $\chi(\nu) = \lambda(\nu)$ *for* $\nu = 1, 2, \ldots, N$.

PROOF. The theorem is proved by induction on N. It is certainly true for $N = 1, 2, 3$. Let k be a fixed positive integer greater than 3 and suppose we know that the theorem is true for $N = 1, 2, \ldots$ $k-1$ and all admissible functions χ. We shall prove it true for $N=k$ by means of induction on the number of primes p such that $p \leqslant k$ and $\chi(p) \geqslant 0$. If there are no such primes, then $\chi(\nu) = \lambda(\nu)$ for $\nu = 1, 2, \ldots, N$ and there is nothing to prove. Let r be a positive integer and suppose our theorem is true with $N = k$ for those admissible functions X having the property that there are at most $r-1$ primes p with $p \leqslant k$ and $\chi(p) \geqslant 0$. Suppose χ is an admissible function such that there are exactly r primes p with $p \leqslant k$ and $\chi(p) \geqslant 0$, and let q be the largest of these primes. We must prove that

$$\sum_{\nu=1}^{k} \frac{\chi(\nu)}{\nu} > \sum_{\nu=1}^{k} \frac{\lambda(\nu)}{\nu}.$$

We define an admissible function X by the equality

$$X(\nu) = (-1)^{a_\nu} \chi(\nu/q^{a_\nu}),$$

where q^{a_ν} is the highest power of q dividing ν. Then $\chi(q) \geqslant 0$, $X(q) = -1$ whereas $X(p) = \chi(p)$ for any prime p different from q. Hence there are exactly $r-1$ primes p such that $p \leqslant k$ and $X(p) \geqslant 0$. Since our theorem holds for $N \leqslant k-1$ and all admissible functions and since it holds when $N = k$ for the admissible function X, we have

$$\sum_{\nu=1}^{k}\frac{\chi(\nu)}{\nu}=\sum_{q\nmid\nu,\,\nu\leqslant k}\frac{\chi(\nu)}{\nu}+\sum_{\nu\leqslant k/q}\frac{\chi(q\nu)}{q\nu}$$

$$=\sum_{q\nmid\nu,\,\nu\leqslant k}\frac{X(\nu)}{\nu}+\frac{\chi(q)}{q}\sum_{\nu\leqslant k/q}\frac{\chi(\nu)}{\nu}$$

$$>\sum_{q\nmid\nu,\,\nu\leqslant k}\frac{X(\nu)}{\nu}$$

$$>\sum_{q\nmid\nu,\,\nu\leqslant k}\frac{X(\nu)}{\nu}+\frac{X(q)}{q}\sum_{\nu\leqslant k/q}\frac{X(\nu)}{\nu}$$

$$=\sum_{\nu=1}^{k}\frac{X(\nu)}{\nu}\geqslant\sum_{\nu=1}^{k}\frac{\lambda(\nu)}{\nu}.$$

This completes the proof of the theorem.

It is clear from the proof that we can weaken condition (iii) for admissibility to read "either $\chi(p)\geqslant 0$ or $\chi(p)=-1$". It does not seem immediately obvious, however, that we can weaken it to "$\chi(p)\geqslant -1$" or even to "$-1\leqslant\chi(p)\leqslant 1$".

Turán [6] reported that Conjecture I has been verified for all $n\leqslant 1000$. Showalter's theorem then shows that Conjecture II is true for all $n\leqslant 1000$ and all real characters (in fact, all admissible arithmetic functions).

Of course for a fixed real character χ the inequality $\sum_{\nu=1}^{n}\chi(\nu)/\nu>0$ holds if n is large enough, since $\sum_{\nu=1}^{\infty}\chi(\nu)/\nu>0$. We can obtain a more specific assertion as follows. We may assume that χ is a real non-principal character modulo D. Then if $S(\nu)=\sum_{\mu=n+1}^{\nu}\chi(\mu)$, it is known that $|S(\nu)|<\frac{5}{3}D^{1/2}\log D$ for any ν greater than n (cf. [5], p. 34, and [3], pp. 85-86). Hence

$$\sum_{\nu=1}^{n}\frac{\chi(\nu)}{\nu}=\sum_{\nu=1}^{\infty}\frac{\chi(\nu)}{\nu}-\sum_{\nu=n+1}^{\infty}\frac{S(\nu)-S(\nu-1)}{\nu}$$

$$= \sum_{\nu=1}^{\infty} \frac{\chi(\nu)}{\nu} - \sum_{\nu=n+1}^{\infty} \frac{S(\nu)}{\nu(\nu+1)}$$

$$> \sum_{\nu=1}^{\infty} \frac{\chi(\nu)}{\nu} - \frac{5}{3} \frac{D^{1/2} \log D}{n+1}.$$

Now it is known that $\sum_{\nu=1}^{\infty} \chi(\nu)/\nu > D^{-\epsilon/2}$ for D sufficiently large (cf. [1]) and that

$$\sum_{\nu=1}^{\infty} \chi(\nu)/\nu > D^{-1/2} \log \frac{1}{2}(3+\sqrt{5})$$

$$> \frac{20}{21} D^{-1/2},$$

for all D. Combining these facts, we find that

$$\sum_{\nu=1}^{n} \chi(\nu)/\nu > 0,$$

for $n > D^{(1/2)+\epsilon}$, provided D is sufficiently large, and for $n > \frac{7}{4} D \log D$, with no restriction on D.

In conjunction with Satz, 454 of [4], the formula

$$\frac{\zeta(2s)}{\zeta(s)(s-1)} = \int_1^{\infty} \left(\sum_{\nu \leq x} \frac{\lambda(\nu)}{\nu} \right) \frac{dx}{x^s},$$

shows that Conjecture I implies the Riemann hypothesis, as was pointed out by Turán [6]. Indeed Ingham [2] has pointed out that Conjecture I implies not only that the complex zeros of $\zeta(s)$ all have real part $\frac{1}{2}$, but also that they are all simple and that their imaginary parts satisfy infinitely many linear relations with rational coefficients.

Recently Otto Larsen has reported that he has verified Conjecture I for all $n \leq 4528$ [*Mat. Tidsskr.* B (1952), 28]. Also the Institute for Numerical Analysis in Los Angeles has verified it for all $n < 100{,}000$, but apparently no report of this has yet been published.

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University of Illinois
and
University of Colorado

A NOTE ON THE LINEAR SPACE GENERATED BY A SEQUENCE OF INTEGRAL FUNCTIONS

By V. GANAPATHY IYER

[Received July 31, 1953]

1. Let $\alpha = \Sigma a_n z^n$ and $\beta = \Sigma b_n z^n$ be two integral functions, $\alpha \circ \beta = \Sigma a_n b_n z^n$ and let $(\alpha)_n$ denote $\alpha \circ \alpha \ldots \circ \alpha$, n times. In a recent paper [1], I have proved the following result.

THEOREM A. *Let the integral function* $\alpha = \Sigma a_n z^n$ *satisfy the following conditions* :

(*i*) *the real part of* a_n *decreases as* n *increases* ;

(*ii*) *no* a_n *is zero and no two* a_n*'s are equal.*

Then every integral function could be obtained as the uniform limit, in any finite circle, of finite linear combinations of $(\alpha)_n$, $n = 1, 2, \ldots$

2. The object of this note is to show that *the conclusion of Theorem* A *is true without the hypothesis* (*i*), *that is under* (*ii*) *alone.* The proof of the theorem consisted in showing that if (e_n) be a sequence with $|e_n|^{1/n}$ bounded and $\Sigma e_n a_n \exp [a_n z]$ is identically zero, then $e_n a_n = 0$ and therefore $e_n = 0$ assuming (*i*) and (*ii*). I shall now show that this conclusion is true under (*ii*) alone. This result is obviously contained in the following result.

THEOREM B. *Let* Σa_n *and* Σb_n *be two absolutely convergent series. Let* $g(z) = \Sigma b_n \exp [a_n z]$ *be identically zero. If the* a_n*'s satisfy* (*ii*) *of Theorem* A, *then* $b_n = 0$, $n = 0, 1, 2, \ldots$.

PROOF. Let $A(z) = \sum_{1}^{\infty} b_{p_i} \exp [a_{p_i} z]$, $B(z) = \sum_{1}^{\infty} b_{q_i} \exp [a_{q_i} z]$ and $C(z) = \sum_{1}^{\infty} b_{r_i} \exp [a_{r_i} z]$ denote the sums of the terms of $g(z)$ such that the corresponding a_n's have their real part greater than, equal to and less than zero respectively. Since $a_n \to 0$ as $n \to \infty$, we can

suppose the indices p_i so arranged that the real parts of the a_{p_i}'s decrease. In view of absolute convergence, the convergence or sum of the series concerned will be unaffected. If $t > 0$, we get

$$b_{p_1} t + \sum_2 \frac{b_{p_i}[\exp\{(a_{p_i} - a_{p_1})t\} - 1]}{a_{p_i} - a_{p_1}} +$$
$$+ \int_0^t [B(z) + C(z)] \exp[-a_{p_1} z]\, dz = 0. \quad (1)$$

If $x \neq 0$ and real part of $x \geqslant 0$, we see that $|\int_0^t e^{-xz} dz| \leqslant 2/|x|$. Using this, the fact that no two a_n's are equal, and that $[B(z) + C(z)]$ is bounded in $0 \leqslant z < \infty$, we see that all the terms in the left side of (1) except $b_{p_1} t$ remain bounded as $t \to \infty$. Hence $b_{p_1} = 0$. Repeating this argument we get, by induction, that all the $b_{p_i} = 0$. Similarly taking $t < 0$, we conclude that all $b_{r_i} = 0$. In $B(z)$, now identically zero, there is no need to suppose any particular ordering of the terms. The previous arguments remain valid whichever term of $B(z)$ is taken. We get

$$b_{q_1} t + \sum_2^\infty \frac{b_{q_i}[\exp\{(a_{q_i} - a_{q_1})t\} - 1]}{a_{q_i} - a_{q_1}} = 0.$$

Since $|[\exp\{(a_{q_i} - a_{q_1})t\} - 1]| \leqslant 2$ and $\sum_2^\infty b_{q_i}/\{(a_{q_i} - a_{q_1})\}$ is absolutey convergent we get $b_{q_1} = 0$. This completes the proof the theorem.

3. Remarks. *If* $b_n = a_n$, the assumption that no two a_n's are equal can be dropped, since the equal terms in $g(z)$ can be grouped together. Thus we see that if $\sum_0^\infty a_n$ is absolutely convergent and $\sum_0^\infty a_n \exp a_n z$ is identically zero, then $a_n = 0$ for $n = 0, 1, 2, \ldots$. This result is equivalent to the statement that if all the series $\sum_0^\infty a_n^k$ converge absolutely to the sum zero, $k = 1, 2, \ldots$ then $a_n = 0$ for $n = 0, 1, 2, \ldots$. It has been recently proved by K. Zeller [2] that the result is no longer true without the hypothesis of absolute convergence.

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